DIVERGENT SERIES JUST GOT MORE CONVERGENT

Eric A. Galapon
National Institute of Physics
University of the Philippines
Diliman, Quezon City

Samahang Pisika ng Pilipinas
2015 National Conference
University of Northern Philippines
Vigan, Ilocos Sur

Convergent series

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\ldots=\frac{\pi^2}{6}$$

Divergent series

$$\sum_{k=0}^{\infty}k^2=1+4+9+16+\ldots=\infty$$

"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever"

Neil Henrik Abel

$$1 + 2! + 3! + 4! + \ldots = \infty$$

But physics is replete with divergent expansions arising from perturbative solutions to the differential equations of mathematical physics.

Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r})+V(\vec{r})\psi(\vec{r})=E\psi(\vec{r})$$

Perturbative solution diverges in
the asymptotic regime $\vec{r}\to\infty$

What good there is in a divergent series?

“Divergent series converge faster than convergent series because they don't have to converge.”

George F. Carrier

The incomplete gamma function

$$\Gamma(v, x) = \int_x^\infty t^{v-1} e^{-t}\, \mathrm{d}t$$

assume

$$v > 0 \text{ and } n \neq 1, 2, 3, \ldots$$

Expansion-1

$$\Gamma(v,x) = \int_x^\infty t^{v-1} e^{-t}\,\mathrm{d}t$$

$$= \int_0^\infty t^{v-1} e^{-t}\,\mathrm{d}t - \int_0^x t^{v-1} e^{-t}\,\mathrm{d}t$$

$$= \Gamma(v) - \int_0^x \mathrm{d}t\; t^{v-1} \sum_{k=0}^\infty \frac{(-1)^k t^k}{k!}$$

$$= \Gamma(v) - x^v \sum_{k=0}^\infty \frac{(-x)^k}{k!(v+k)}$$

Expansion-1

$$\Gamma(v,x) = \Gamma(v) - x^{v} \underbrace{\sum_{k=0}^{\infty} \frac{(-x)^{k}}{k!(v+k)}}_{\text{Convergent series}}$$

Expansion-2

$$\Gamma(v,x)=\int_x^{\infty} t^{v-1}e^{-t}\,\mathrm{d}t$$

$$=e^{-x}x^{v-1}-(1-v)\int_x^{\infty} e^{-t}t^{v-2}\,\mathrm{d}t$$

$$=e^{-x}x^{v-1}-(1-v)e^{-x}x^{v-2}+(1-v)(2-v)\int_x^{\infty} e^{-t}t^{v-3}\,\mathrm{d}t$$

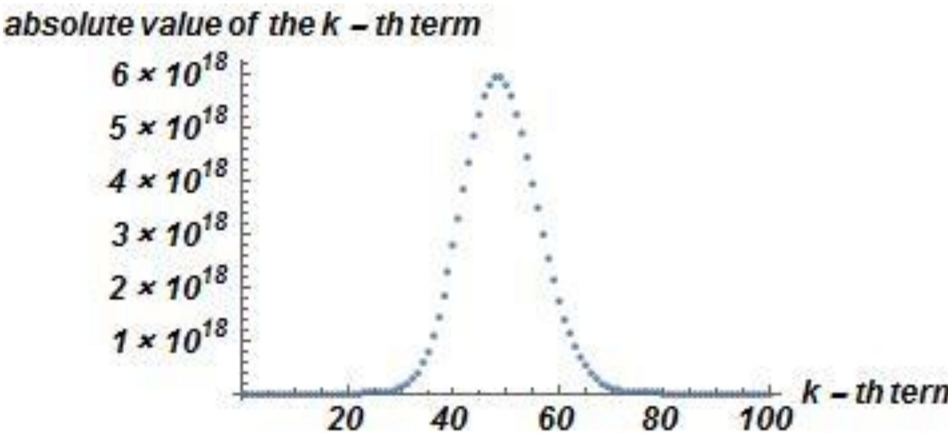
$$=e^{-x}x^{v-1}\sum_{k=0}^{\infty}(1-v)_k\,\frac{(-1)^k}{x^k}$$

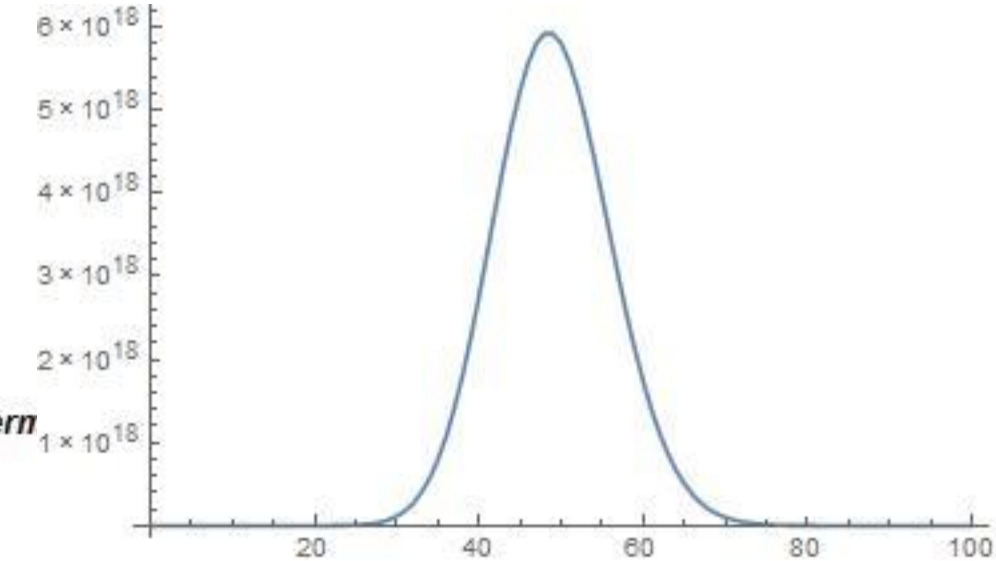
Expansion-2

$$\Gamma(v,x) = e^{-x} x^{v-1} \sum_{k=0}^{\infty} (1-v)_k \frac{(-1)^k}{x^k}$$

Divergent series

x	50		
v	1/3.		
exact	**1.4027670218856X10^(-23)**		
N	**convergent series**		**divergent series**
1	-8.373155961		1.4211150385793*10^(-23)
2	129.7780252		1.4021668380649*10^(-23)
3	-1843.810278		1.4027984447488*10^(-23)
4	21181.38659		1.4027647590590*10^(-23)
5	-200214.7371		1.4027672293429*10^(-23)
50	1.08475381429189*10^19		1.4027670218856*10^(-23)
100	2.08055859827124 2*10^10		1.4027670218855*10^(-23)
110	1.11876886976597 3*10^7		1.4027670217446*10^(-23)
120	2440.509653		1.4027664658255*10^(-23)
180	1.4079711899470*10^(-23)		-0.124387698
200	1.4027670218856*10^(-23)		-45485156450


absolute value of the k – th term
6 × 10^18
5 × 10^18
4 × 10^18
3 × 10^18
2 × 10^18
1 × 10^18
20
40
60
80
100
k – th tern


6 × 10^18
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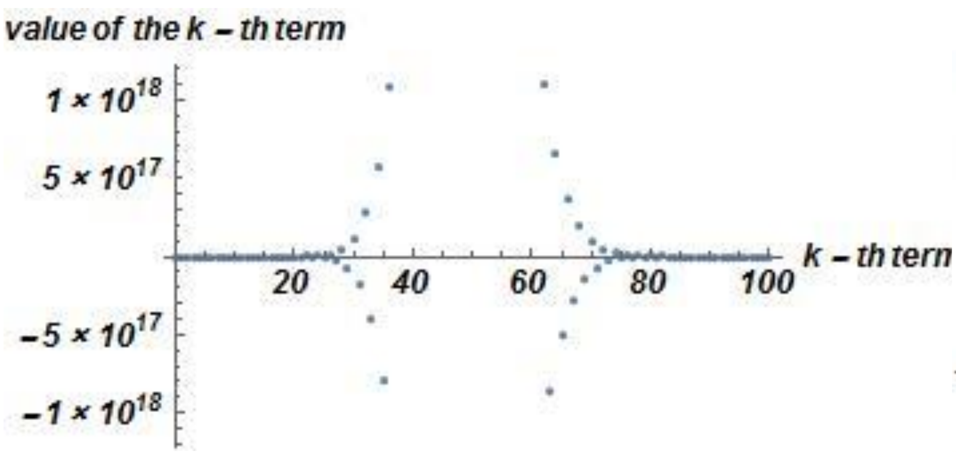
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1 × 10^18
5 × 10^17
−5 × 10^17
−1 × 10^18
20
40
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80
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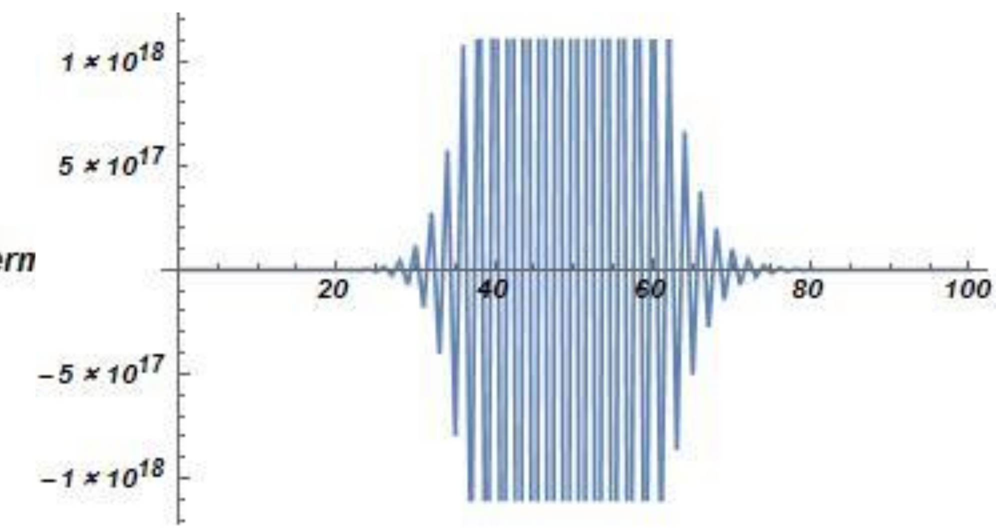
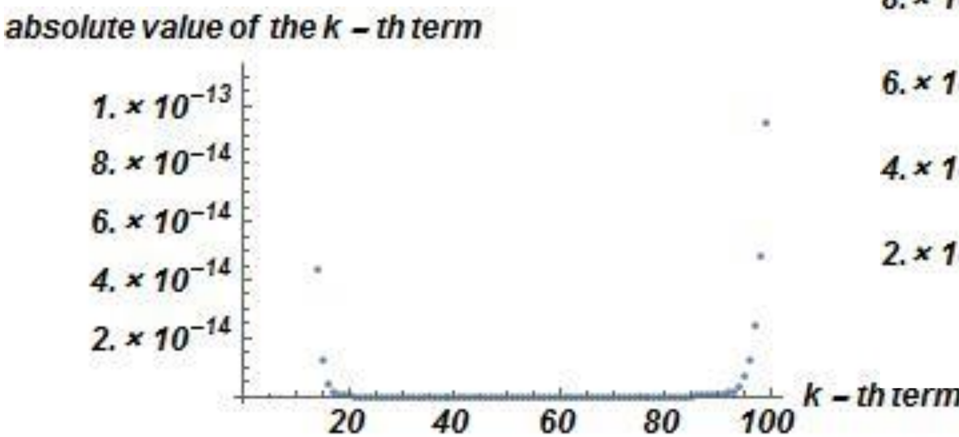
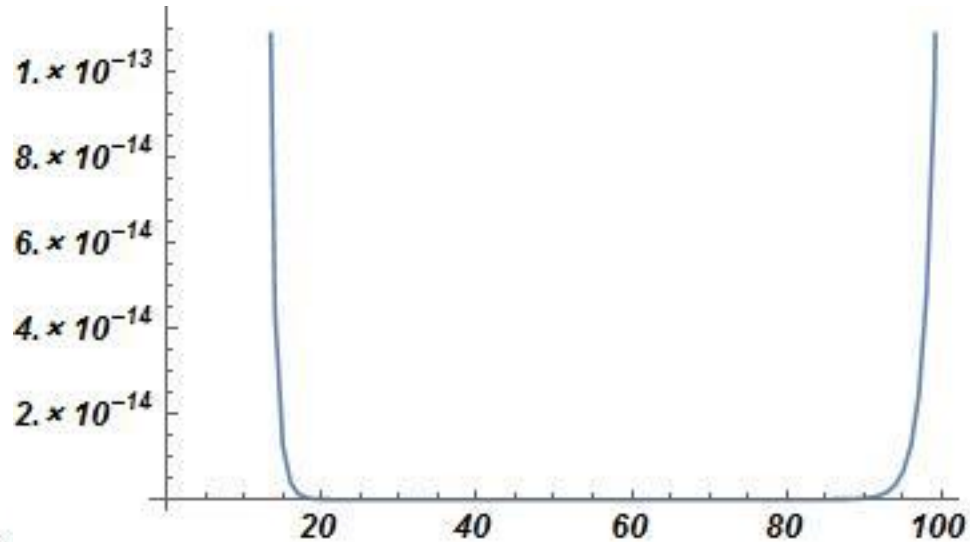
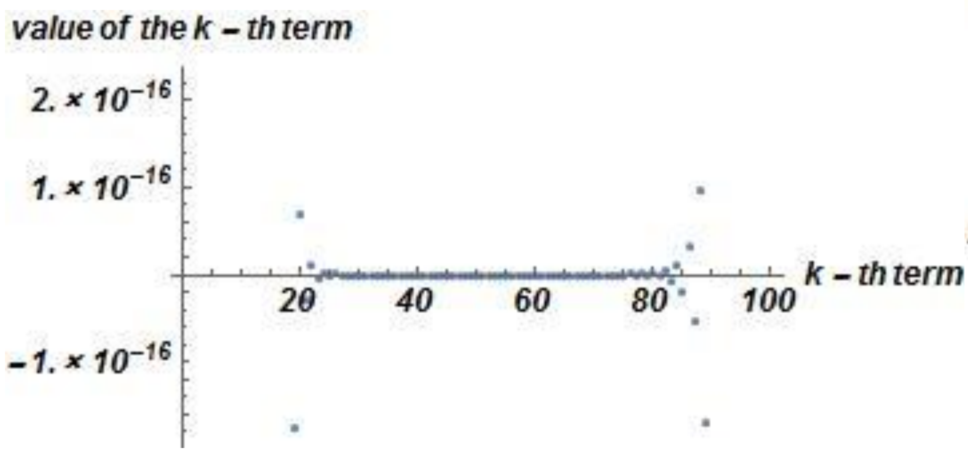
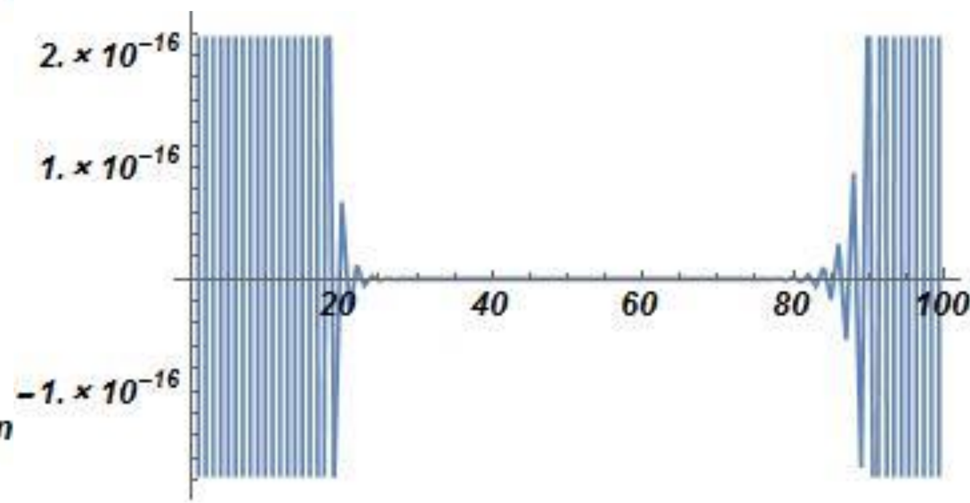
k – th tern


1 × 10^18
5 × 10^17
−5 × 10^17
−1 × 10^18
20
40
60
80
100

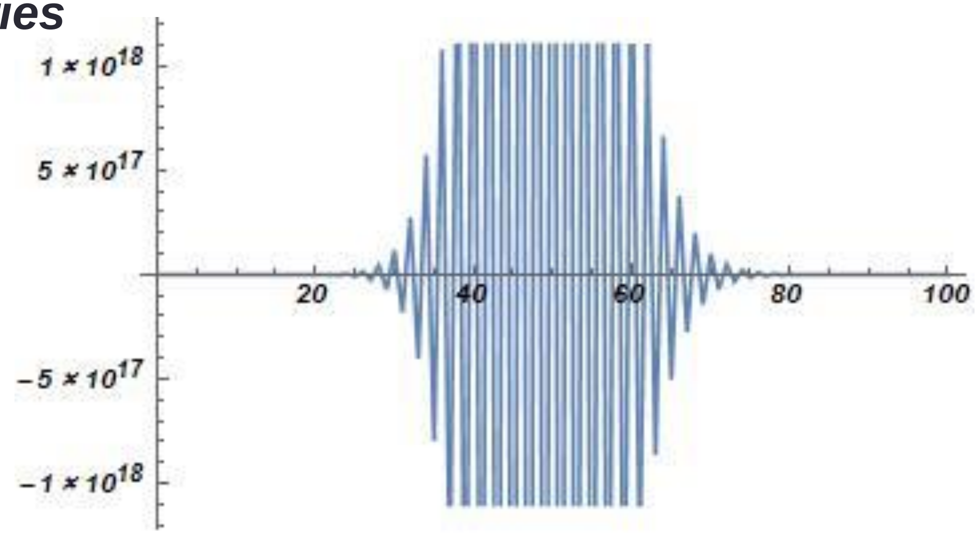

absolute value of the k – th term
1. × 10^−13
8. × 10^−14
6. × 10^−14
4. × 10^−14
2. × 10^−14
20
40
60
80
100
k – th term


1. × 10^−13
8. × 10^−14
6. × 10^−14
4. × 10^−14
2. × 10^−14
20
40
60
80
100

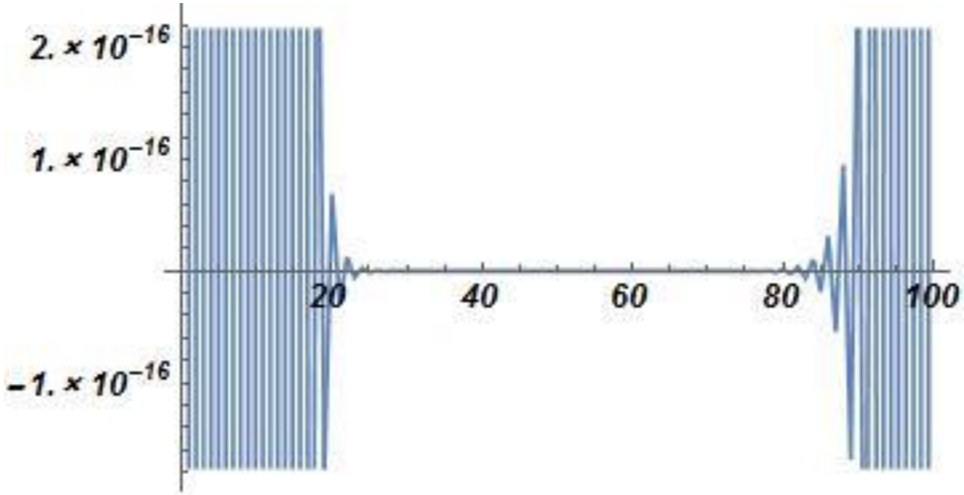

value of the k – th term
2. × 10^−16
1. × 10^−16
20
40
60
80
100
k – th term
−1. × 10^−16


2. × 10^−16
1. × 10^−16
20
40
60
80
100
−1. × 10^−16

The complementary nature of convergent and divergent series



Convergent series



Divergent series

Super-asymptotic summation

The sum of a divergent series up to the least term.

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{x^k} \quad \Rightarrow \quad f_S(x) = \sum_{k=0}^{N} \frac{a_k}{x^k}$$

N is the location of the least term.

Super-asymptotic sum

X=50

V=1/3

$$\Gamma(v,x) = e^{-x} x^{v-1} \sum_{k=0}^{\infty} (1-v)_k \frac{(-1)^k}{x^k}$$

- Exact value

$$1.40276702188560400047313 \times 10^{-23}$$

- Super-asymptotic sum

$$1.40276702188560400047308 \times 10^{-23}$$

- Relative error

~10^(-22)

- Accuracy increases as $x \to \infty$

In the asymptotic regime $x \to \infty$

Convergent series

Slow convergence

Arbitrary accuracy

Divergent series

Rapid convergence

Limited accuracy

But accuracy increases as $x \to \infty$

This explains why time dependent scattering theory is effectively described by the leading term of the divergent perturbative solution of the Schrodinger equation.

$$\psi(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}} + f_k(\theta)\frac{e^{ikr}}{r}, \quad r \to \infty$$

There is a natural barrier to the accuracy of the super-asymptotic sum.

Can we break through the barrier?

Why bother to ask for more accuracy beyond the super-asymptotic sum?

- Super-asymptotic sum is accurate only in the asymptotic or perturbative regime.

- We wish to obtain insight in the non-asymptotic or non-perturbative regime using the perturbative result.

Hyperasymptotics

By M. V. Berry and C. J. Howls
H. H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, U.K.

We develop a technique for systematically reducing the exponentially small ('superasymptotic') remainder of an asymptotic expansion truncated near its least term, for solutions of ordinary differential equations of Schrödinger type where one transition point dominates. This is achieved by repeatedly applying Borel summation to a resurgence formula discovered by Dingle, relating the late to the early terms of the original expansion. The improvements form a nested sequence of asymptotic series truncated at their least terms. Each such 'hyperseries' involves the terms of the original asymptotic series for the particular function being approximated, together with terminating integrals that are universal in form, and is half the length of its predecessor. The hyperasymptotic sequence is therefore finite, and leads to an ultimate approximation whose error is less than the square of the original superasymptotic remainder. The Stokes phenomenon is automatically and exactly incorporated into the scheme. Numerical computations confirm the efficacy of the technique.

M.V. Berry and C.J. Howles, *Hyperasymptotics*, Proc. Roy. Soc. Lond. A, 430, 653-668 (1990)

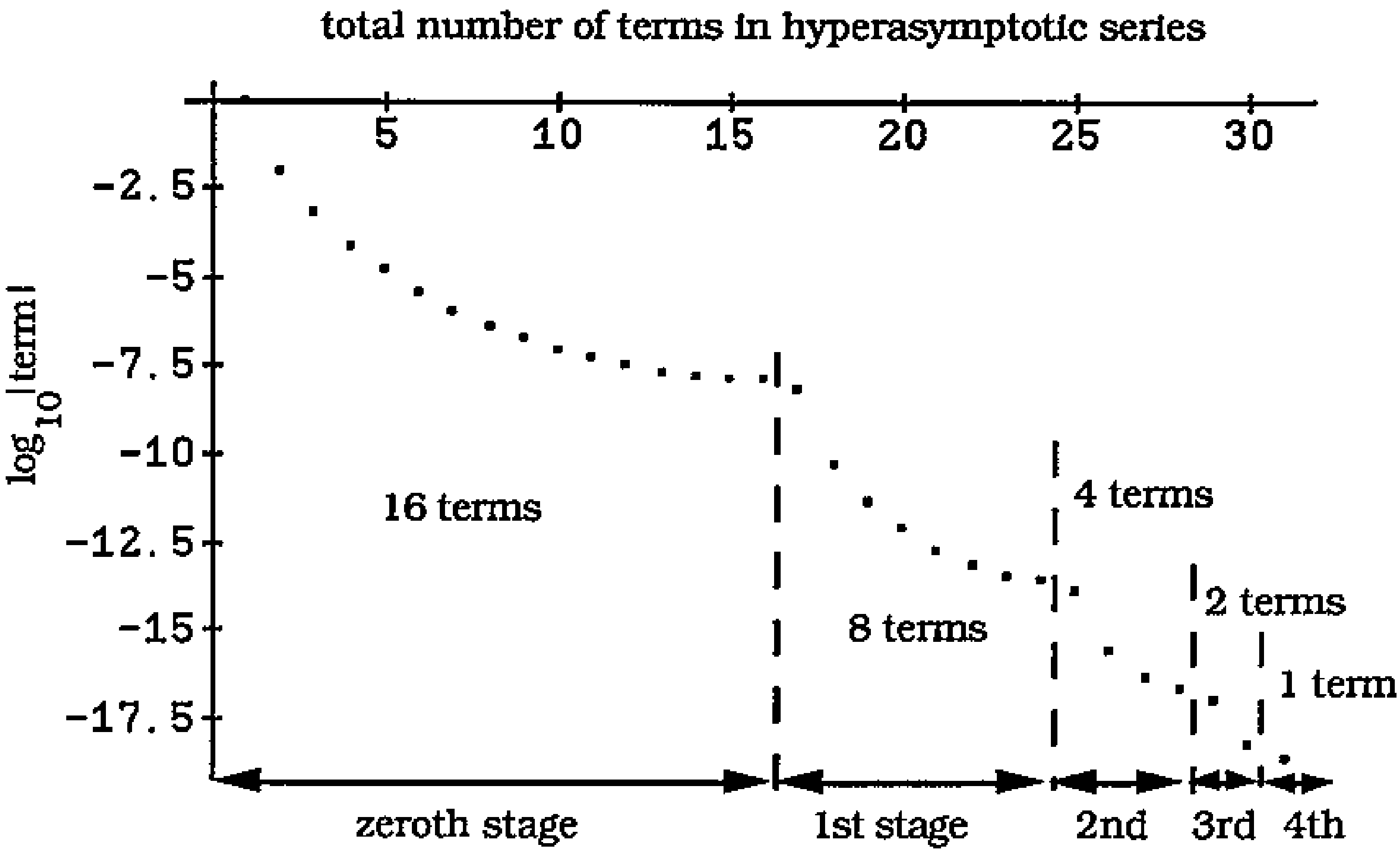
Hyperasymtotics…

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{x^k}$$

$$= \sum_{k=0}^{N_0} \frac{a_k}{x^k} + \sum_{k=N_0+1}^{\infty} \frac{a_k}{x^k}$$

$$= \sum_{k=0}^{N_0} \frac{a_k}{x^k} + \frac{1}{x^{N_0+1}} \sum_{k=0}^{N_1} f_{1,k}(x) + \frac{1}{x^{N_0+1}} \sum_{k=N_1+1}^{\infty} f_{1,k}(x)$$

…re-expansion of the remainder term.



Hyperasymptotics breaks the super-asymptotic barrier.

M.V. Berry and C.J. Howles, *Hyperasymptotics*, Proc. Roy. Soc. Lond. A, 430, 653-668 (1990)

Limitations

- Hyperasyptotics has a natural barrier to its accuracy.

- Hyperasymptotics is computationally expensive.

Going beyond hyperasymptotic accuracy

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Cite this article: Galapon EA, Martinez KML. 2014 Exactification of the Poincaré asymptotic expansion of the Hankel integral: spectacularly accurate asymptotic expansions and non-asymptotic scales. *Proc. R. Soc. A* **470**: 20130529.
http://dx.doi.org/10.1098/rspa.2013.0529

Exactification of the Poincaré asymptotic expansion of the Hankel integral: spectacularly accurate asymptotic expansions and non-asymptotic scales

Eric A. Galapon[1] and Kay Marie L. Martinez[1,2]

[1]Theoretical Physics Group, National Institute of Physics, University of the Philippines, Diliman Quezon City, 1101 The Philippines
[2]Siliman University, Dumaguete City, 6200 The Philippines

We obtain an exactification of the Poincaré asymptotic

The Paris integral

$$F_\nu(x) = \int_0^\infty \frac{J_\nu(xt)}{1+t}\,\mathrm{d}t, \quad \nu \neq -1, \quad x > 0$$

Poincare asymptotic expansion

$$F_{P,\nu}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma((\nu+k+1)/2)}{\Gamma((\nu-k+1)/2)} \frac{2^k}{x^{k+1}}, \quad x \to \infty$$

A resummation of the Poincare asymptotic expansion

$$F_{R,v}(x) = \csc(v\pi)\sum_{s=0}^{\infty}\frac{1}{(2x)^{s+\frac{1}{2}}}\frac{\left(\frac{1}{2}-v\right)_s\left(\frac{1}{2}+v\right)_s}{s!\left(\frac{1}{2}\right)_s}$$

$$\times\left[e^{i\frac{\pi}{2}(s+v-\frac{3}{2})}e^{-ix}\Gamma\left(\frac{1}{2}+s,-ix\right)+e^{-i\frac{\pi}{2}(s+v-\frac{3}{2})}e^{ix}\Gamma\left(\frac{1}{2}+s,ix\right)\right],$$

Comparison of the accuracy of Poincare, resummation and hyperasymptotic summation

Relative Error					
x	Poincare	Resummed (I G F)	Hyperasymptotic		
			1st level	2nd level	3rd level
10	4.0×10^{-5}	1.1×10^{-9}	3.0×10^{-6}	2.5×10^{-7}	2.8×10^{-7}
15	2.3×10^{-6}	3.1×10^{-13}	7.5×10^{-9}	7.8×10^{-13}	7.0×10^{-13}
20	3.0×10^{-8}	1.3×10^{-17}	3.3×10^{-11}	1.5×10^{-16}	1.5×10^{-16}
25	1.8×10^{-9}	1.5×10^{-21}	3.1×10^{-12}	1.5×10^{-20}	1.4×10^{-20}
30	2.5×10^{-11}	8.8×10^{-26}	6.5×10^{-16}	1.2×10^{-26}	1.2×10^{-26}
35	1.5×10^{-11}	3.1×10^{-30}	1.3×10^{-19}	3.5×10^{-28}	3.5×10^{-28}
40	2.2×10^{-14}	7.4×10^{-34}	3.7×10^{-22}	1.1×10^{-30}	2.0×10^{-30}
45	1.3×10^{-15}	1.0×10^{-38}	3.5×10^{-25}	3.1×10^{-32}	2.0×10^{-32}
50	1.9×10^{-17}	7.7×10^{-42}	1.6×10^{-27}	1.2×10^{-38}	3.3×10^{-38}

Divergent series just got more convergent!

Table from R.J. Fajardo, MS Physics Thesis, NIP-UPD 2015.

The Poincare and resummed series are formally equivalent.

$$F_{P,\nu}(x) \Leftrightarrow F_{R,\nu}(x)$$

$$F_{P,\nu}(x) \xrightarrow{\text{transformation}} F_{R,\nu}(x)$$

$$F_{R,\nu}(x) \xrightarrow{\text{transformation}} F_{P,\nu}(x)$$

But they are not numerically equivalent.

A sequence of functions

$$\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$$

Is called an ***asymptotic sequence*** if

$$\frac{\phi_{k+1}(x)}{\phi_k(x)} \to 0 \quad \text{as} \quad x \to \infty$$

The sequence is also called an ***asymptotic scale***.

Example:

$$\{x^{-1}, x^{-2}, x^{-3}, \ldots\}$$

The entire theory of asymptotics is based on asymptotic scales.

But the resummed series are expansions in non-asymptotic scales.

The resummation in non-asymptotic sequence

$$F_{R,\nu}(x) = \csc(\nu\pi)\sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}-\nu\right)_s\left(\frac{1}{2}+\nu\right)_s}{s!\left(\frac{1}{2}\right)_s}\Psi_s(x)$$

$$\Psi_s(x) = \frac{1}{(2x)^{s+\frac{1}{2}}}\times\left[e^{i\frac{\pi}{2}(s+\nu-\frac{3}{2})}e^{-ix}\Gamma\left(\frac{1}{2}+s,-ix\right)+e^{-i\frac{\pi}{2}(s+\nu-\frac{3}{2})}e^{ix}\Gamma\left(\frac{1}{2}+s,ix\right)\right],$$

$$\lim_{x\to\infty}\frac{\Psi_{s+j}(x)}{\Psi_s(x)} = 1, \qquad j = 1,2,\dots\ ,\ s = 0,1,\dots$$

Non-asymptotic sequence

Conclusions

- A given Poincare series can be resummed once to obtain accuracy beyond hyperasymptotic summation.

- Non-asymptotic scales provide meaningful and accurate asymptotic sum.

Open problems

- What is the theory behind the most accurate resummation for a given Poincare asymptotic expansion?
- How should the theory of asymptotics be rewritten in terms of non-asymptotic scales?
- What specific areas of physics benefit from the theory of asymptotic expansions by non-asymptotic scales?

Students working on these problems

Raiseth John Fajardo

Anton Hilado

Join them!

quant-math.org/wp/

The research is funded by the University of the Philippines System Grants through ECWRG-2015-2-016 with project title "Asymptotic expansions by non-asymptotic scales."