Exactification of the Poincare asymptotic expansion of the Hankel integral: Spectacularly accurate asymptotic expansions and non-asymptotic scales

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Abstract

We obtain an exactification of the Poincare asymptotic expansion of the Hankel integral, $\int_0^\infty f(x)J_\nu(bx)\mathrm{d}x$ as $b\to\infty$, using the distributional approach of Mc Clure and Wong [*J. Inst. Maths Applics* **3**, pp 129-145, 1978]. We find that, for half-integer orders of the Bessel function, the exactified asymptotic series terminates, so that it gives an exact finite sum representation of the Hankel integral. For other orders, the asymptotic series does not terminate and is generally divergent, but is amenable to superasymptotic summation, that is, by optimal truncation. For specific examples, we compare the accuracy of the optimally truncated asymptotic series due to the Mc Clure-Wong distributional method to that of the optimally truncated asymptotic series due to the Mellin-Barnes integral method. We find that the former is spectacularly more accurate than the latter, by, in some cases, more than seventy (70) orders of magnitude for the same moderate value of b. Moreover, the exactification can lead to a resummation of the Poincare asymptotic expansion when it is exact, with the resummmed Poincare series exhibiting again the same spectacular accuracy. More importantly, the distributional method may yield meaningful resummations that involve scales that are not asymptotic sequences.

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1 Introduction

The Hankel integral, $\int_0^\infty f(x)J_\nu(bx)\mathrm{d}x$, arises naturally in many fields of application in physics [1, 2, 3, 4, 5]. A recent example, which has served as the motivation behind this work, arises in quantum tunneling problem where the traversal time across a potential barrier appears in the form of a Hankel integral of the zeroth order [5]. In many occasions it is desirable to obtain an asymptotic estimate of the integral for arbitrarily large b [1, 5]. It is well known that, if $f(x) = \Phi(x)$ is infinitely differentiable at the origin, the Hankel integral has the Poincare asymptotic expansion (PAE) [6, 8, 9, 10, 11]

$$\int_0^\infty \Phi(x)J_\nu(bx)\mathrm{d}x \sim \frac{1}{2}\sum_{s=0}^{\infty}\frac{\Phi^{(s)}(0)}{s!}\frac{\Gamma\left(\frac{1}{2}(\nu+s+1)\right)}{\Gamma\left(\frac{1}{2}(\nu-s+1)\right)}\left(\frac{2}{b}\right)^{s+1}. \quad b\to\infty. \tag{1}$$

However, there is a large class of functions in the domain of equation 1 for which equation 1 itself fails to give a meaningful information on the asymptotic behavior of the integral for arbitrarily large b. This may happen, for example, when the integral is exponentially small in b, so that it is beyond all orders and equation 1 reduces to $\int_0^\infty \Phi(x)J_\nu(bx)\mathrm{d}x \sim 0+0+0+\dots$ as $b\to\infty$. A well-known example of this case is the class of integrals $I(\omega) = \int_0^\infty e^{-x^2}g(x^2)xJ_0(\omega x)\mathrm{d}x$ which arises from high energy nuclear physics [6, 12, 13, 14, 15]. Also for integer values of ν, equation 1 terminates, giving the impression that the integral reduces to a polynomial in inverse power of b in the asymptotic limit. However, in general, there are trailing exponentially small terms that are not detected by the Poincare asymptotic expansion 1.

Various methods have been developed in obtaining the asymptotic behaviors and explicit asymptotic expansions of Hankel integrals that are outside the range of the Poincare asymptotic expansion 1. For example, Frenzen and Wong [6, 14] used the method of shifting the contour of integration in the complex plane (now a standard method in exponential asymptotics [16]) to investigate and obtain the asymptotic behavior of the class of Hankel integrals $I(\omega)$. They showed that the asymptotic behavior of $I(\omega)$ depends on the analytic properties of the function $f(z)$ in the complex plane. For $f(z)$ having algebraic and logarithmic singularities, they developed a method to obtain the explicit asymptotic expansion. When $f(z)$ is holomorphic in the upper complex plane, they obtained asymptotic expansion with a trailing exponentially small term whose order is only given. But when $f(z)$ is an entire function they only obtained an order estimate, so that no explicit asymptotic expansion was available. Gabutti and Lepora [15] latter tackled the same integral and its generalization given by the integral $\int_0^\infty f(x^2)J_\nu(\omega)x^{\nu+1}\mathrm{d}x$. Their method, which is based on the Laplace transform, gives explicit asymptotic expansions for a limited class of exponentially small Hankel integrals. A powerful method of obtaining explicit asymptotic expansion of a Hankel integral is the Mellin-Barnes method [7], which is a modification of the classical Mellin transform method [6, 17, 18] based on the asymptotic expansion of the ratio of products of gamma functions [19, 20, 21, 22, 23]. This method is capable of obtaining explicit expansions where the method of Franzen and Wong can only give order estimates. In fact, asymptotic expansions of the examples of the present paper can be obtained through the Mellin-Barnes method and will be compared with our results.

However, the above methods are at most prescriptive, providing a means of obtaining the asymptotic expansion of a given specific Hankel integral. What we wish to see is a sufficiently general single expression that includes the dominant and sub-dominant contributions. In Dingle's terms [25], we wish to obtain an exact asymptotic expansion for the Hankel integral or to exactify the Poincare asymptotic expansion 1 [26, 27]. In this paper, we wish to obtain the exactification of 1 for the Hankel integral

$$I(b) = \int_0^\infty \Phi(x) J_\nu(bx) dx, \tag{2}$$

for arbitrary real ν and arbitrary function $\Phi(x)$ in the Schwartz space $\mathcal{S}(\mathbb{R}^+)$ and positive b. We accomplish this by means of the distributional approach in asymptotics developed by McClure and Wong [6, 28], which was first used in deriving explicit error terms for the Stieltjes transform [28], and then latter applied to the Fourier transform [29], and extended to a large class of integral transforms [9]. Here we will directly use the original results of McClure and Wong in obtaining the asymptotic expansion of the Hankel integral $I(b)$. We will obtain the result

$$\int_0^\infty \Phi(x) J_\nu(bx) \mathrm{d}x = \frac{1}{2} \sum_{s=0}^\infty \frac{\Phi^{(s)}(0)}{s!} \frac{\Gamma\left(\frac{1}{2}(\nu+s+1)\right)}{\Gamma\left(\frac{1}{2}(\nu-s+1)\right)} \left(\frac{2}{b}\right)^{s+1} + \Delta_\nu(b), b \to \infty, \tag{3}$$

where the first term is the known Poincare asymptotic expansion 1 applied to the $\Phi(x)$'s; and the second term is the sub-dominant correction to the standard Poincare expansion, carrying the exponentially small or beyond-all-order parts of the Hankel integral. The equality in equation 3 indicates that no sub-dominant term is dropped.

The exactified Poincare asymptotic expansion 3 will lead us to consider formal series representation of a function $F(z)$ of the form $F(z) = \sum_{k=0}^\infty a_n \varphi_n(z)$ as $z \to \infty$, where the a_n's are constants independent of z and the $\varphi_n(z)$'s comprise a sequence of functions of z. In this paper, we will refer to the sequence $\{\varphi_n(z)\}$ as a scale for the function $f(z)$ or simply a scale. Also we will refer to a scale of power-type if $\varphi_{n+j}(z)/\varphi_n(z) = z^{-\lambda_{n+j}+\lambda_n}$ for some fixed λ_n and $j = 1, 2, \ldots$, otherwise, non-power-type. The scale $\{\varphi_n(z)\}$ is referred to as an asymptotic sequence [6] or asymptotic scale [7] if $\varphi_{n+1}(z) = o(\varphi_n(z))$ as $z \to \infty$. In [24] a scale satisfying the conditions $\varphi_{n+1}(z) = O(\varphi_n(z))$ and $\varphi_{n+2}(z) = o(\varphi_n(z))$ as $z \to \infty$ is referred to as a semiasymptotic sequence. Here equation 3 will impose on us to admit scales that are neither asymptotic nor semiasymptotic scale. This clearly muddies the definition of generalized asymptotic series, whose foundation rests on asymptotic scales [7] or its recent suggested further generalization [24], as the formal series $F(z) = \sum_{k=0}^\infty a_n \varphi_n(z)$ is no longer necessarily an asymptotic series in the generalized Poincare sense

[6, 7]. It is not our objective here to introduce a general theory of asymptotic series that goes beyond the generalized Poincare series. It is our goal to motivate its further generalization by working out the consequences of the exactified Hankel integral. For our present purposes, we will refer to a formal divergent series as an asymptotic series if it manifests the behavior of initial convergence, followed by eventual divergence with increasing number of terms in the series for sufficiently large values of the asymptotic parameter.

A given function may be expanded in different scales, potentially in infinitely many ways. Two such expansions will be referred to as transformations of each other if one can formally manipulate one to assume the other. For the Hankel integral, we will find that the McClure-Wong method leads to an asymptotic series in a scale other than the asymptotic scale due to the Mellin-Barnes method. The latter is generally expressed in non-power-type scales, such as the special functions of mathematical physics; the former is expressed in power-type scales. We will demonstrate that the two expansions are formally equivalent by showing that the McClure-Wong expansion can be reduced to the Mellin-Barnes expansion; that is, the two expansions are transformations of each other. However, their formal equivalence lies only on the surface—beneath is a great gulf separating them in approximating the numerical value of the Hankel integral. We will demonstrate that the asymptotic series due to McClure-Wong method possesses a spectacular accuracy in approximating the value of the Hankel integral. Moreover, even in cases where the exactifying term vanishes, the distributional method yields an expansion expressed in mixed scales. One of the terms is in power-type scale of the Poincare asymptotic expansion; the other, in non-power-type scale. We will demonstrate that these two terms can be combined in either one of the scales. Again we will find that the expansion in non-power-type scale due to the distributional approach yields a much more accurate approximation of the integral than that of the asymptotic expansion in power-type scale of the Poincare series. More important, we will find that the scale is not an asymptotic or a semiasymptotic scale. These results can be seen as a specific generalization of the known transformations of the Poincare asymptotic expansions first considered in detail in [30] and further explored in [31, 32, 33, 34, 35].

The paper is organized as follows. In Section-2, we give a brief review of McClure and Wong's distributional approach; we limit our discussion to what is relevant to the problem at hand. In Section-3, we apply the distributional method to the Hankel integral to obtain an asymptotic expansion. In Section-4, we consider different special cases of the application of the exactified Poincare asymptotic expansion, and give specific examples. In Section-5, we demonstrate the spectacular numerical accuracy of the asymptotic expansion due to the McClure-Wong method compared to the asymptotic expansion due to the Mellin-Barnes method. In Section-6, we reconsider a known Hankel integral which has a Poincare asymptotic expansion which is exact. We show how the distributional method allows one to resum the series in a specific scale. We will find that the Hankel integral is rewritten in a scale which is not asymptotic but numerically meaningful. In Section-7, we conclude and discuss briefly the implications of our results to hyperasymptotics and the general theory of asymptotic expansions.

2 The Distributional Approach in Asymptotics

The basic idea in obtaining the asymptotic expansion of $I(b)$ for $b \to \infty$ is to substitute the asymptotic expansion of the Bessel function $J_\nu(bx)$ for large b back into the integral 2, and then perform the required integration. However, this leads to the divergent integrals $\int_0^\infty e^{\pm ibx}\Phi(x)x^{-k-\frac{1}{2}}\,\mathrm{d}x$ for positive integer k. One may remedy this by using analytic extension theory, i.e., by restricting k to values that the integral exists and then extending the result to the divergent integral by substituting k with positive integer values. But naive application of analytic extension theory has been demonstrated to yield incomplete asymptotic expansions—some terms are missing in the expansion [6].

The distributional approach of McClure and Wong gives the proper treatment and interpretation to such divergent integrals. Their idea is to interpret the integrands as distributions and the integral as functionals on them. Only after such identifications that analytic extension can be applied on divergent integrals arising from the method. We will show that applying the distributional approach to our divergent integrals give explicit asymptotic expansions for $I(b)$. In the following we summarize the distributional method. We restrict the summary to what is relevant to the Hankel integral. We refer the reader to [6] for a thorough discussion of the distributional method.

Consider a locally integrable function $f(x)$ on the interval $[0, \infty)$ with polynomial growth at infinity. We now give the distributional meaning to the integral $\int_0^\infty \Phi(x)f(x)\mathrm{d}x$, where $\Phi(x)$ is an arbitrary function in the Schwartz space $\mathcal{S}$. For each $n \geq 1$, let $f(x)$ have the asymptotic expansion

$$f(x) = \sum_{s=0}^{n-1} a_s e^{icx}x^{-s-\alpha} + f_n(x), \quad x \to \infty \tag{4}$$

where $0 < \alpha < 1$ and c is real. Let $e_s(x) = e^{icx}x^{-s-\alpha}$. From [28], the functions $f(x)$, $e_s(x)$, and $f_n(x)$ generate distributions on $\mathcal{S}$ defined as follows

$$\langle f, \Phi\rangle = \int_0^\infty f(x)\Phi(x)dx \tag{5}$$

$$\langle e_s, \Phi\rangle = \frac{1}{s!}\int_0^\infty \Phi^{(s+1)}(x)\int_x^\infty (\tau - x)^s e_s(\tau)d\tau dx \tag{6}$$

$$\langle f_n, \Phi\rangle = (-1)^n \int_0^\infty f_{n,n}(x)\Phi^{(n)}(x)dx = R_n. \tag{7}$$

where $f_{n,n} = \frac{(-1)^n}{(n-1)!}\int_x^\infty (\tau - x)^{n-1} f_n(\tau)\,\mathrm{d}\tau$ and R_n is the remainder term.

The usefulness of these distributions rests on their exact relationship. They are related according to

$$\langle f, \Phi\rangle = \sum_{s=0}^{n-1} a_s\langle e_s, \Phi\rangle - \sum_{s=0}^{n-1} b_s\langle \delta^{(s)}, \Phi\rangle + \langle f_n, \Phi\rangle \tag{8}$$

where

$$b_s = \frac{(-1)^{s+1}}{s!}\left[M[f;s+1] - \sum_{k=0}^{s-1} a_k \frac{\Gamma(s-k-\alpha+1)}{(c/i)^{s-k-\alpha+1}}\right] \tag{9}$$

and $\delta^{(s)}$ is the s-th derivative of the Dirac delta function at the origin, and $M[f;z]$ is the Mellin transform or its analytic extension when the integral diverges.

3 Asymptotic Expansion of The Hankel Integral

We now apply the distributional approach to the Hankel integral 2 for large values of parameter b. In order to obtain an expansion in the form of Eq.(4), we write the Bessel function in terms of the Hankel functions using the relationship $2J_\nu(z) = H_\nu^{(1)}(z) + H_\nu^{(2)}(z)$ [36]. The Hankel integral 2 then splits in two integrals, $I(b) = I_1(b) + I_2(b)$, with $I_l(b) = \frac{1}{2}\int_0^\infty \Phi(x) H_\nu^{(l)}(bx)\,\mathrm{d}x, \quad l = 1, 2$. We will apply the distributional approach to the integrals $I_l(b)$ with the identifications $f^{(l)}(x) = H_\nu^{(l)}(bx)$.

Using the known asymptotic expansions of the Hankel functions for large argument [36], we have the following asymptotic expansions for $f^{(k)}(x)$

$$f^{(l)}(x) = H_\nu^{(l)}(bx) = \sum_{s=0}^{n-1} a_s^{(l)} e^{(-1)^{l+1} ibx} x^{-s-\frac{1}{2}} + f_n^{(l)}(x),\ x \to \infty, \tag{10}$$

where the coefficients are given by

$$a_s^{(l)} = \sqrt{\frac{2}{\pi}}\frac{1}{2^s s!}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{\Gamma\left(\nu-k+\frac{1}{2}\right)} e^{(-1)^{l+1} i\frac{\pi}{2}(s-\nu-\frac{1}{2})}\frac{1}{b^{s+\frac{1}{2}}}, \tag{11}$$

for $l = 1, 2$.

Now let $e_s^{(l)}(x) = e^{(-1)^{l+1} ibx} x^{-s-\frac{1}{2}}$ and define the distributions according to equations 5 to 7. We have to consider different distributions corresponding to the two Hankel functions separately. The full expansion for the integral $I(b)$ is obtained by adding these distributions according to

$$I(b) = \frac{1}{2}\left[\sum_{s=0}^{\infty} a_s^{(1)}\langle e_s^{(1)}, \Phi\rangle + a_s^{(2)}\langle e_s^{(2)}, \Phi\rangle - (-1)^s \Phi^{(s)}(0) B_s\right] \tag{12}$$

where $B_s = b_s^{(1)} + b_s^{(2)}$, and $b_s^{(l)}$, for $l = 1$ and $l = 2$, is obtained from equation 9 by substituting the function f with $H_\nu^{(l)}$; we have already extended the summation to infinity in equation 12. Equation (6) can be further simplified by interchanging the order of integration and performing integration-by-parts to the inner integral. We then have

$$\langle e_s^{(l)}, \Phi\rangle = I_s^{(l)} - \sum_{k=0}^{s} \frac{\Phi^{(s-k)}(0)}{(s-k)!}\frac{\Gamma(1/2-k)}{((-1)^l ib)^{1/2-k}} \tag{13}$$

where we have let

$$I_s^{(l)}(b) = \int_0^\infty e^{(-1)^{l+1} ib\tau} \tau^{-s-1/2} \Phi(\tau) d\tau. \tag{14}$$

The integrals $I_s^{(l)}$, which are Mellin transforms, are generally divergent, and they have to be understood as analytic extensions of their convergent versions. For the Mellin transform in Eq.(9), we note that $f^{(1)}(x) + f^{(2)}(x) = 2J_\nu(bx)$. Then, from [6, Lemma 4, p203] we have

$$M[f^{(1)} + f^{(2)}; s+1] = \frac{\Gamma[\frac{1}{2}(\nu+s+1)]}{\Gamma[\frac{1}{2}(\nu-s+1)]} \left(\frac{2}{b}\right)^{s+1}. \tag{15}$$

Substituting equations 13, 14 and 15 back into equation 12 and performing simplifications, we obtain our main result given by equation 3, where the exactifying term is given by

$$\begin{aligned} \Delta_\nu(b) \;=\; & -\sum_{s=0}^{\infty} (-1)^s \frac{\Phi^{(s)}(0)}{s!} \frac{\Gamma\left(\frac{1}{2}(\nu+s+1)\right)}{\Gamma\left(\frac{1}{2}(\nu-s+1)\right)} \\ & \times \cos\left(\frac{\pi}{2}(s-\nu)\right) \sin\left(\frac{\pi}{2}(\nu+s+1)\right) \left(\frac{2}{b}\right)^{s+1} \\ & + \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{\infty} \frac{\Gamma\left(\nu+s+\frac{1}{2}\right)}{2^s\, s!\, \Gamma\left(\nu-s+\frac{1}{2}\right) b^{s+1/2}} \left[e^{i\frac{\pi}{2}(s-\nu-1/2)} I_s^{(1)}(b) + e^{-i\frac{\pi}{2}(s-\nu-1/2)} I_s^{(2)}(b)\right]. \end{aligned} \tag{16}$$

We can further simplify equation 3 by expanding the trigonometric factors in the first term of equation 16. We find that the Poincare term gets canceled out, leaving the expression

$$\begin{aligned} I(b) \;=\; & -\frac{\cos(\pi\nu)}{2} \sum_{s=0}^{\infty} \frac{(-1)^s \Phi^{(s)}(0)}{s!} \frac{\Gamma[\frac{1}{2}(\nu+s+1)]}{\Gamma[\frac{1}{2}(\nu-s+1)]} \left(\frac{2}{b}\right)^{s+1} \\ & + \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{\infty} \frac{\Gamma\left(\nu+s+\frac{1}{2}\right)}{2^s\, s!\, \Gamma\left(\nu-s+\frac{1}{2}\right) b^{s+1/2}} \left[e^{i\frac{\pi}{2}(s-\nu-1/2)} I_s^{(1)}(b) + e^{-i\frac{\pi}{2}(s-\nu-1/2)} I_s^{(2)}(b)\right]. \end{aligned} \tag{17}$$

It is this form of the asymptotic expansion that we will use in the Sections to follow.

Observe that expansion 3 has two distinct terms. The first term is the standard Poincare asymptotic expansion of the Hankel integral as $b \to \infty$. This expansion can be obtained using the classical methods, such as the summability and Mellin transform methods, and those in [8, 9, 10, 11]. These methods completely miss out the second term. In general the first term (when it is not identically zero) is the dominant term and the second term is the sub-dominant term in the asymptotic expansion. The presence of sub-dominant term in the expansion allows us to view equation 17 as the completion of the Poincare expansion in the sense of Dingle [25]: An asymptotic series is said to be complete when it contains all dominant and sub-dominant asymptotic components. The process of obtaining the sub-dominant terms in and adding these to the standard Poincare expansion is also called exactification [26, 27]. The asymptotic expansion 17 is then the exactification of the Poincare asymptotic expansion for the Hankel integral.

The above results can be extended in two fronts: First, we can extend the above treatment for $f(x) = \Phi(x)x^{-\lambda}$, where $\Phi(x)$ belongs to $\mathcal{S}(\mathbb{R}^+)$, $1 > \lambda > 0$, $(\nu - \lambda) > -1$, and $\Phi(0) \neq 0$. The only required modification is the identification $f^{(k)}(x) = x^{-\lambda} H_\nu^k(bx)$. The relevant distributions are $e^{\pm ibx} x^{-s-\lambda-\frac{1}{2}}$. For $\lambda \neq 1/2$, we have the same definitions as in Section-2; for $\lambda = 1/2$, we have a special case which can be treated separately [6]. Second, the application of McClure and Wong's distributional approach is not limited to the Hankel integral. It may be applied to the general class of integral transforms $\int_0^\infty \Phi(x) f(bx) \mathrm{d}x$, where $f(x)$ has the asymptotic expansion given by equation 4 where α can now be in the interval $(0, 1]$ and c can now be zero. The application of the distributional approach for such an integral proceeds as in the Hankel integral: Expanding $f(bx)$ asymptotically as $b \to \infty$ inside the integral, followed by identifying each integral term that appears after the expansion as a distribution, and finally applying the appropriate relationship among the distributions [6].

4 Special Cases and Examples

4.1 Finite series representation of the Hankel integral

When $\nu = n + \frac{1}{2}$, with $n = 0, 1, \ldots$, the first term of the expansion 17 vanishes and the second term terminates at the term $s = n$, yielding

$$I(b) = \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{n} \frac{1}{2^s \, s!} \frac{(n+s)!}{(n-s)!} \frac{1}{b^{s+\frac{1}{2}}} \left[e^{i\frac{\pi}{2}(s-n-1)} I_s^{(1)}(b) + e^{-i\frac{\pi}{2}(s-n-1)} I_s^{(2)}(b) \right] \tag{18}$$

Since the the asymptotic series 17 terminates and 17 itself is exact, equation 18 is then equal to the integral itself. Equation 18 provides a finite series representation of the Hankel integral for half-integer orders of the Bessel function.

4.1.1 Example

Let us consider the Hankel integral

$$h_n(b) = \int_0^\infty e^{-x} J_{n+\frac{1}{2}}(bx) \, \mathrm{d}x = \frac{b^n \sqrt{b}}{\sqrt{b^2+1} \left(\sqrt{b^2+1}+1\right)^n \sqrt{\sqrt{b^2+1}+1}}, \quad b > 0, \tag{19}$$

which can be obtained by a term by term integration of the series expansion of the Bessel function and then summing the resulting series. The integrals $I_s^{(1)}(b) = I_s^{(2)^*}(b) = \int_0^\infty e^{(ib-1)x} x^{-s-\frac{1}{2}} \, \mathrm{d}x$ converge for $s = 0$ and diverge for all $s > 0$. We assign values to them by means of the analytic extension of the integral

$$\int_0^\infty e^{(ib-1)x} x^{-\lambda-\frac{1}{2}} \mathrm{d}x = (1-ib)^{-\frac{1}{2}+\lambda}\, \Gamma\left(\frac{1}{2}-\lambda\right), \quad \mathrm{Re}\,\lambda < \frac{1}{2},\ b \in \mathbb{R}, \tag{20}$$

which is obtained by a simple change of variable in the complex plane.

With the substitution $\lambda = s$ in the right hand side of equation 20, we obtain the values of $I_s^{(1)}(b)$ and $I_s^{(2)}(b)$. Substituting these integrals back into equation 18, we arrive at

$$\begin{aligned} h_n(b) &= \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{n} \frac{1}{2^s\, s!} \frac{(n+s)!}{(n-s)!} \frac{\Gamma\left(\frac{1}{2}-s\right)}{b^{s+\frac{1}{2}}} \\ &\quad \times \left[e^{i\frac{\pi}{2}(s-n-1)}(1-ib)^{s-\frac{1}{2}} + e^{-i\frac{\pi}{2}(s-n-1)}(1+ib)^{s-\frac{1}{2}}\right]. \end{aligned} \tag{21}$$

The equality of equations 19 and 21 can be established with the help of the identity

$$\sqrt{1+ib} = \frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{b^2+1}+1} + i\sqrt{\sqrt{b^2+1}-1}\right], \quad b > 0, \tag{22}$$

with $\sqrt{1-ib}$ obtained from equation 22 by complex conjugation. This example illustrates how the distributional method, in particular equation 18, may yield a finite series representation of the value of a Hankel integral.

4.2 Beyond all orders Hankel integrals

When $\nu = 0$, odd terms in the Poincare asymptotic expansion vanish by virtue of the factor $1/\Gamma((1-s)/2)$, and only even terms contribute. Hence when $\Phi(x)$ is odd in x, $\Phi(-x) = -\Phi(x)$, the even terms also vanish, and all terms of the Poincare asymptotic expansion entirely vanish. The same is true when the derivative of $\Phi(x)$ of all orders vanish at the origin, $\Phi^{(s)}(0) = 0$. Under these conditions, the first term of equation 17 also vanish, so that the asymptotic expansion 17 reduces to

$$I(b) = \frac{1}{\sqrt{2\pi}} \sum_{s=0}^{\infty} \frac{(-1)^s \left[\left(\frac{1}{2}\right)_s\right]^2}{2^s\, s!\, b^{s+1/2}} \left[e^{i\frac{\pi}{2}(s-1/2)} I_s^{(1)}(b) + e^{-i\frac{\pi}{2}(s-1/2)} I_s^{(2)}(b)\right]. \tag{23}$$

The vanishing of the Poincare asymptotic expansion implies that the Hankel integral is beyond all orders or exponentially small. The class of integrals $I(\omega) = \int_0^\infty e^{-x^2} g(x^2) x J_0(\omega x) \mathrm{d}x$ belongs to this case.

4.2.1 Example

Let us consider the Hankel integral

$$I_1(b) = \int_0^\infty e^{-cx^2} \sin(ax) J_0(bx) dx \tag{24}$$

for real a, b and c. This integral is proportional to the "effective index of refraction" of a Gaussian wave-packet incident upon a barrier potential[5]. To obtain the asymptotic expansion of $I_1(b)$ for large b, let us instead obtain the asymptotic expansion of the integral $\tilde{I}_1(b) = \int_0^\infty e^{-cx^2+iax} J_0(bx)dx$. Integral 24 is obtained by taking the imaginary part of the integral $\tilde{I}_1(b)$. To evaluate the integrals $I_s^{(1,2)}$, we use the integral identity [37, p365, #3.462(1)],

$$\int_0^\infty x^{\lambda-1} e^{-\beta x^2 - \gamma x} \mathrm{d}x = (2\beta)^{-\lambda/2}\Gamma(\lambda)\exp(\gamma^2/8\beta)D_{-\lambda}(\gamma/\sqrt{2\beta}) \qquad (25)$$

for $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\beta) > 0$, where $D_\rho(z)$ is the parabolic cylinder function (PCF). The integrals $I_s^{(1,2)}$ are then obtained by analytically extending 25 for negative values of λ. We then have

$$I_s^{(1,2)} = \frac{\Gamma(1/2-s)}{(2c)^{1/4-s/2}} e^{-(a\pm b)^2/8c} D_{s-1/2}\left[\frac{-i(a\pm b)}{\sqrt{2c}}\right], \qquad (26)$$

in which the upper (lower) sign corresponds to I_1 (I_2).

We next evaluate $\Phi^{(s)}(0)$ for $\Phi(x) = e^{-cx^2+ibx}$. Using the known Rodriguez representation of the Hermite polynomials, $H_n(z) = (-1)^n e^{z^2} \partial_z^2 e^{-z^2}$, we get

$$\Phi^{(s)}(0) = \sum_{r=0}^{\lfloor s/2 \rfloor} \begin{pmatrix} s \\ 2r \end{pmatrix} c^r H_{2r}(0)(ia)^{s-2r} \qquad (27)$$

where $H_{2r}(0) = (-1)^2(2r)!/r!$ is a Hermite polynomial evaluated at zero, and $\lfloor z \rfloor$ is the integer part. For $\nu = 0$ we are particularly interested in the even derivatives of the test function $\Phi(x)$. Plugging these results into equation 17 gives us a full expansion for the Hankel integral $\tilde{I}_1(b)$. For $I_1(b)$, we are interested in the imaginary part so we use the functional identity of the PCF in [37, p1030, #9.248(1)]

$$D_{-p-1}(-iz) = e^{i\pi(-p-1)/2}\Gamma(-p)[D_p(z) - e^{i\pi p}D_p(-z)]/\sqrt{2\pi} \qquad (28)$$

Then, since $\Phi^{(2s)}(0)$ is real for all s, the imaginary part of $\tilde{I}_1(b)$, which is the integral we seek, is given by

$$I_1(b) = \sum_{s=0}^\infty \frac{(-1)^s\left[\left(\frac{1}{2}\right)\right]^2(\sqrt{2c})^{s-\frac{1}{2}}}{2^{s+1}\, s!\, b^{s+\frac{1}{2}}}\left[\frac{D_{-s-\frac{1}{2}}\left(\frac{b-a}{\sqrt{2c}}\right)}{e^{(b-a)^2/8c}} - \frac{D_{-s-\frac{1}{2}}\left(\frac{b+a}{\sqrt{2c}}\right)}{e^{(b+a)^2/8c}}\right]. \qquad (29)$$

Observe that equation 29 is expanded in terms of the non-power-type scale given by

$$\phi_s(b) = \frac{1}{b^{s+\frac{1}{2}}}\left[e^{-(b-a)^2/8c}D_{-s-1/2}\left(\frac{b-a}{\sqrt{2c}}\right) - e^{-(b+a)^2/8c}D_{-s-1/2}\left(\frac{b+a}{\sqrt{2c}}\right)\right] \qquad (30)$$

$s = 0, 1, 2, \ldots$ Using the known asymptotic expansion for the parabolic cylinder function, it can be shown that this sequence of functions satisfies $\phi_{s+1}(b)/\phi_s(b) = O(b^{-2})$, as $b \to \infty$ so that $\phi_{s+1}(b) = o(\phi_s(b))$ as $b \to \infty$; that is, sequence 30 is an asymptotic sequence. We can consider the sequence $\phi_s(b)$ as a compound sequence and consider each term separately, which are on their own non-power type scales. We find that each is an asymptotic sequence.

We now wish to obtain the asymptotic expansion for $I_1(b)$ in power-type scale. We accomplish this by using the asymptotic expansion of $D_\nu(z)$ given by

$$D_\nu(z) = z^\nu e^{-z^2/4} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{-\nu}{2})_k (\frac{1-\nu}{2})_k 2^k}{k! z^{2k}}, \quad |z| \to \infty. \tag{31}$$

for $|\arg z| < 3\pi/4$. Notice that we have used the equality sign in 31 instead of the usual asymptotic symbol $\sim$ to indicate that no sub-dominant term is dropped in the expansion, i.e. it is complete in the sense of Dingle [25]. By expanding the resulting $(b-a)^{-s-2k-1/2}$ and $(b+a)^{-s-2k-1/2}$ terms binomially for large b and collecting terms in powers of b, we get

$$\begin{aligned} I_1(b) \quad &= \quad \frac{1}{2b} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{m!} \, {}_3F_1\left[\begin{matrix} -m, -\frac{m}{2}, \frac{1}{2} - \frac{m}{2} \\ \frac{1}{2} - m \end{matrix} ; -\frac{4c}{a^2} \right] \left(\frac{a}{b}\right)^m \\ &\quad \times \left[e^{-(b-a)^2/4c} - (-1)^m e^{-(b+a)^2/4c} \right] \end{aligned} \tag{32}$$

where ${}_3F_1$ is a specific hypergeometric function. Equation 32 is in compound power type scale. Equation 32 is in compound power type scale. The exponential smallness of $I_1(b)$ for large b is even more evident in this form.

To compare with the Mellin-Barnes method, we obtain the first few terms of equation 32 using the following explicit values of the hypergeometric function involved,

$$\begin{aligned} m = 0: &\qquad {}_3F_1\left[\begin{matrix} 0, 0, \frac{1}{2} \\ \frac{1}{2} \end{matrix} ; z \right] = 1 \\ m = 1: &\qquad {}_3F_1\left[\begin{matrix} -1, -\frac{1}{2}, 0 \\ -\frac{1}{2} \end{matrix} ; z \right] = 1 \\ m = 2: &\qquad {}_3F_1\left[\begin{matrix} -2, -1, -\frac{1}{2} \\ -\frac{3}{2} \end{matrix} ; z \right] = 1 + \frac{2}{3} z \\ m = 3: &\qquad {}_3F_1\left[\begin{matrix} -3, -\frac{3}{2}, -1 \\ -\frac{5}{2} \end{matrix} ; z \right] = 1 + \frac{9}{5} z \end{aligned}$$

Substituting these values back into equation 32, we obtain the expansion

$$\begin{aligned} I_1(b) \quad &= \quad e^{-(a^2+b^2)/4c} \sinh\left(\frac{ab}{2c}\right) \left[\frac{1}{b} + \frac{3}{8}\left(a^2 - \frac{8}{3}c\right) \frac{1}{b^3} + \ldots \right] \\ &\quad + e^{-(a^2+b^2)/4c} \cosh\left(\frac{ab}{2c}\right) \left[\frac{a}{2}\frac{1}{b^2} + \frac{5}{16}\left(a^3 - \frac{36}{5}ac\right) \frac{1}{b^4} + \ldots \right] \end{aligned} \tag{33}$$

This result can be obtained using the Mellin-Barnes method. (See Appendix.)

4.3 Terminating Poincare asymptotic expansions

When the order, ν, of the Bessel function is any positive integer, the Poincare asymptotic expansion terminates, and the distributional method gives a trailing beyond all order correction terms. In particular, for $\nu = 2n$ and $\Phi^{(2j)}(0) = 0$ (such as when $\Phi(x)$ is odd in x) for $j = 0, 1, 2, \ldots$, equation 17 becomes

$$I(b) = \frac{1}{2}\sum_{j=0}^{n-1} \frac{\Phi^{(2j+1)}(0)}{(2j+1)!}\frac{(n+j)!}{(n+j-1)!} \tag{34}$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{s=0}^{\infty}\frac{\Gamma\left(2n+s+\frac{1}{2}\right)}{2^s\, s!\,\Gamma\left(2n-s+\frac{1}{2}\right)b^{s+1/2}}\left[e^{i\frac{\pi}{2}(s-2n-\frac{1}{2})}I_s^{(1)}(b) + e^{-i\frac{\pi}{2}(s-2n-\frac{1}{2})}I_s^{(2)}(b)\right].$$

Also for $\nu = 2m+1$, $m = 0, 1, 2, \ldots$ and $\Phi^{(2j+1)}(0) = 0$, $j = 0, 1, 2, \ldots$ (such as when $\Phi(x)$ is even in x),

$$I(b) = \frac{1}{2}\sum_{j=0}^{m} \frac{\Phi(2j)(0)}{(2j)!}\frac{(m+j)!}{(m-j)!} \tag{35}$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{s=0}^{\infty}\frac{\Gamma\left(2m+s+\frac{3}{2}\right)}{2^s\, s!\,\Gamma\left(2m-s+\frac{3}{2}\right)b^{s+1/2}}\left[e^{i\frac{\pi}{2}(s-2m-\frac{3}{2})}I_s^{(1)}(b) + e^{i\frac{\pi}{2}(s-2m-\frac{3}{2})}I_s^{(2)}(b)\right].$$

The first terms of equations 34 and 35 are the finite sum Poincare asymptotic expansions, and the second terms are the trailing beyond all orders corrections.

4.3.1 Example

Let us consider the integral

$$I_2(b) = \int_0^\infty e^{-cx^2}\sin(ax)J_{2n}(bx)\,\mathrm{d}x, \tag{36}$$

for $n = 1, 2, \ldots$. We identify $\Phi(x) = e^{-cx^2}\sin(ax)$, which is odd in x. Since the order of the Bessel function is even, equation 35 holds for this case. Following the same steps above, we arrive at the expansion

$$I_2(b) = 2a\sum_{r=0}^{n-1}\frac{(16c)^r\Gamma[n+r+1]U[-r,\frac{3}{2},-\frac{a^2}{4c}]}{(2r+1)!\Gamma[n-r]b^{2r+2}} + \tag{37}$$
$$\frac{(-1)^n}{2\sqrt{c}}\sum_{s=0}^{\infty}\frac{(-1)^s(1/2-2n)_s(1/2+2n)_s}{s!\left(b\sqrt{2/c}\right)^{s+1/2}}\times$$
$$\left[e^{-(b-a)^2/8c}D_{-s-1/2}\left(\frac{b-a}{\sqrt{2c}}\right) - e^{-(b+a)^2/8c}D_{-s-1/2}\left(\frac{b+a}{\sqrt{2c}}\right)\right],$$

where $U(a, b, z)$ is the tricomi confluent hypergeometric function. Observe that the correction to the Poincare asymptotic expansion is expressed in the same asymptotic sequence as in the previous example.

For comparison latter, we rewrite the exponentially small second term of equation 37 in power-type scale of the asymptotic variable b. Again expanding the parabolic cylinder function using 31 following the same procedure above, we arrive at

$$\begin{aligned} I_2(b) &= 2a\sum_{s=0}^{n-1}\frac{\Gamma(n+s+1)}{\Gamma(n-s)}\frac{(16c)^s U\left(-s,3/2,-a^2/4c\right)}{(2s+1)!b^{2s+2}} \\ &+\frac{(-1)^n}{2b}\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)_m\left(\frac{a}{b}\right)^m\left[e^{-(b-a)^2/4c}-(-1)^m e^{-(b+a)^2/4c}\right] \\ &\times\sum_{k=0}^{m/2}\frac{\left(-c/a^2\right)^k}{k!(m-2k)!}\,{}_3F_2\left[\begin{matrix}-k,\frac{1}{2}-2n,\frac{1}{2}+2n \\ \frac{1}{2}-m,\frac{1}{2}\end{matrix};1\right] \end{aligned} \tag{38}$$

Equation 38 is also in compound power-type scale. This result can be obtained by the Mellin-Barnes method.

5 Numerical Accuracy of the Distributional Asymptotic Expansion

The distributional approach yields an asymptotic expansion that is formally equivalent to the asymptotic expansion in power-type scale due to the Mellin-Barnes integral method. Here we demonstrate that the asymptotic expansion due to the distributional method gives a much more accurate approximation of the value of the integral than the latter asymptotic expansion.

Let us first look at the general behaviors of the asymptotic expansion 29. Figure-1 (upper box) shows the graph of the magnitude of the terms

$$\lambda_s=\frac{\left[\left(\frac{1}{2}\right)_s\right]^2(\sqrt{2c})^{s-\frac{1}{2}}}{2^{s+1}\,s!\,b^{s+\frac{1}{2}}}\left[\frac{D_{-s-\frac{1}{2}}\left(\frac{b-a}{\sqrt{2c}}\right)}{e^{(b-a)^2/8c}}-\frac{D_{-s-\frac{1}{2}}\left(\frac{b+a}{\sqrt{2c}}\right)}{e^{(b+a)^2/8c}}\right]$$

for the given parameters in Figure-1. The λ_s's decrease initially in magnitude and then gradually increase with s, which is the typical behavior of a divergent asymptotic series. Figure-1 (lower box) shows the value of the partial sums of equation 29 for increasing number of terms in the sum, in particular, the sum $I_1^{(N)}(b)=\sum_{s=0}^{N}\lambda_s$. The partial sum oscillates around the exact value, with the amplitude initially decreasing with increasing number of terms and eventually increasing without bound. The closest approach of the partial sum to the exact value occurs near the least term. The approximation to the integral due to the expansion 29 is obtained by truncating it at the least term, i.e., its superasymptotic sum. The Poincare type expansion 32 is analyzed in similar fashion. The asymptotic expansion 32 is truncated at the least term to obtain its superasymptotic approximation to the value of the integral. To compare the distributional and the Poincare type expansions for the integral $I_2(b)$, we

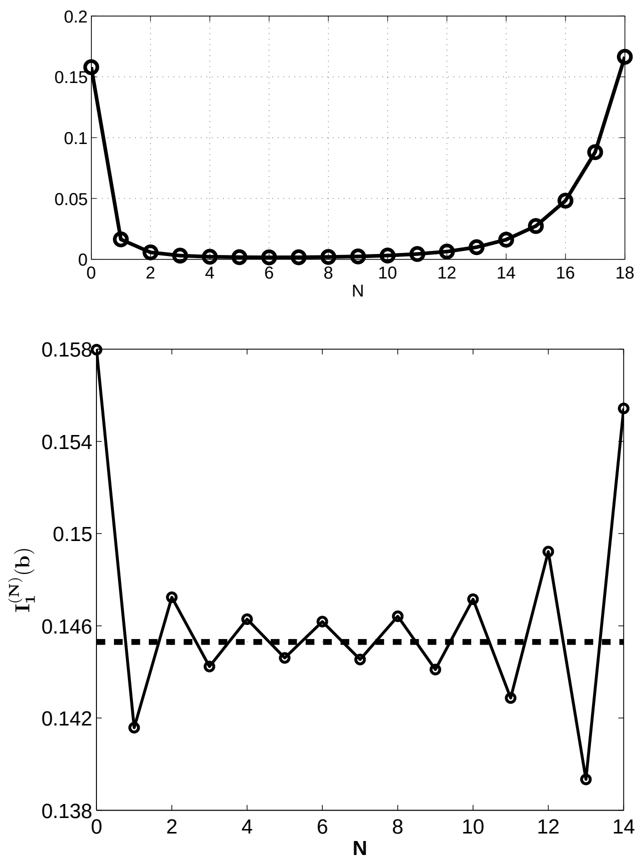

Figure 1: Behavior of $I_1(b)$ with increasing number of terms(N) in the expansion. $a = 1$, $b = 2$, $c = 2$. The exact value of the integral is 0.1453967 and is indicated by the horizontal dashed line.

compare only the correction terms to the truncated Poincare asymptotic expansion. The behavior of the asymptotic expansions are compared in the similar fashion. We find the same behavior of the partial sum of the expansions. Their approximations to the value of the integral are just their optimally truncated versions.

Tables-1 and-2 compare the relative error of the optimally truncated asymptotic expansions due to McClure-Wong method and due to the Mellin-Barnes method for the indicated parameters and for increasing values of the asymptotic parameter b. The exact values are obtained by means of arbitrary precision numerical quadrature using Maple 17. Clearly the optimally truncated distributional expansion is much more accurate than the optimally truncated asymptotic series in power-type scale. Comparing the two, the distributional

	Power-type		Distributional	
b	relative error	least term	relative error	least term
1	8.1×10^{-2}	0	5.6×10^{-2}	3
2	7.8×10^{-2}	0	1.5×10^{-4}	11
3	2.8×10^{-2}	4	1.1×10^{-8}	25
4	8.1×10^{-3}	4	1.8×10^{-14}	45
5	1.0×10^{-3}	5	5.8×10^{-22}	71
6	4.1×10^{-4}	9	3.7×10^{-31}	103
7	1.9×10^{-7}	28	4.4×10^{-42}	141
8	5.8×10^{-10}	30	9.7×10^{-55}	185
9	1.4×10^{-11}	30	4.0×10^{-69}	235
10	1.0×10^{-12}	28	3.1×10^{-85}	291

Table 1: Comparison of the relative error of the optimally truncated power-type asymptotic series and the distributional asymptotic series for the integral $\int_0^\infty e^{-x^2} J_0(bx) \sin x \, \mathrm{d}x$ for different values of the asymptotic parameter b.

takes much more terms before the magnitude of the terms rise again. It is clear that the distributional asymptotic expansion is even effective for relatively small values of b. This accuracy is available provided the two terms (the dominant and sub-dominant) be included simultaneously.

6 Ressummation and Non-asymptotic Sequence

What we wish to do now is to apply the distributional method to an integral with a known non-trivial Poincare asymptotic expansion, and show how a much accurate expansion is obtained from the Poincare asymptotic expansion. Let us consider the integral

$$F_\nu(x) = \int_0^\infty \frac{J_\nu(xt)}{1+t} \, \mathrm{d}t, \quad \nu \neq -1, \quad x > 0. \tag{39}$$

The Poincare asymptotic expansion of this integral as $x \to \infty$ was obtained in [7] using Mellin-Barnes method. The asymptotic expansion is given by

$$F_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{1}{2}(v+k+1)\right)}{\Gamma\left(\frac{1}{2}(v-k+1)\right)} \frac{2^k}{x^{k+1}}, \quad x \to \infty, \tag{40}$$

which differs in [7] by a shift in the index.

Now by identification we have $\Phi(t) = (1+t)^{-1}$ for the integral $F_\nu(x)$. It seems not appropriate to apply the method to this integral because $\Phi(t)$ does not belong to $\mathcal{S}(R^+)$. This is remedied by following Wong [6]. Let $\epsilon > 0$ and introduce the function $\Phi_\epsilon(t) = e^{-\epsilon t}(1+t)^{-1}$. This function belongs to $\mathcal{S}(R^+)$. The original integral is recovered in the limit $\epsilon \to 0$. The result is the direct application of equation 17 to the function $(1+t)^{-1}$.

	Power-type		Distributional	
b	relative error	least term	relative error	least term
1	7.3×10^{0}	6	2.9×10^{-2}	4
2	2.9×10^{-1}	6	9.7×10^{-5}	12
3	3.3×10^{-2}	6	8.5×10^{-9}	26
4	6.3×10^{-3}	6	1.5×10^{-14}	45
5	1.6×10^{-3}	6	5.2×10^{-22}	71
6	2.0×10^{-5}	21	3.4×10^{-31}	103
7	8.1×10^{-7}	21	4.1×10^{-42}	141
8	2.7×10^{-8}	44	9.3×10^{-55}	185
9	1.9×10^{-10}	46	3.9×10^{-69}	235
10	1.5×10^{-12}	46	3.0×10^{-85}	291

Table 2: Comparison of the relative error of the optimally truncated power-type asymptotic series and the distributional asymptotic series for the integral $\int_0^\infty e^{-x^2}\sin(x)J_2(bx)\mathrm{d}x-2U\left(0,\frac{3}{2},-\frac{1}{4}\right)b^{-2}$ for different values of the asymptotic parameter b.

To proceed, we compute for the integrals $I_s^{(1/2)}=\int_0^\infty e^{\pm ixt}t^{-s-1/2}(1+t)^{-1}\mathrm{d}t$ for $s=0,1,\ldots$. Only $s=0$ is convergent; the rest of the integrals diverge and their values will be assigned by analytic continuation of the convergent integral $\int_0^\infty e^{\pm ixt}t^{\lambda-1}(1+t)^{-1}\,\mathrm{d}t$, for $x\in R$ and $0<\operatorname{Re}\lambda<1$. An appropriate rotation of the contour of integration along the complex axis allows us to exploit the following integral identity [37]

$$\int_0^\infty \frac{e^{-\mu t}t^{\nu-1}}{t+\beta}\,\mathrm{d}t=\beta^{\nu-1}e^{\beta\nu}\Gamma(\nu)\Gamma(1-v,\beta\mu), \tag{41}$$

for $|\arg\beta|<\pi$, $\operatorname{Re}\mu>0$, and $\operatorname{Re}\nu>0$, where $\Gamma(a,z)$ is the incomplete gamma function. Appropriate identifications of the parameters of the integral 41 for the integrals $I_s^{(1/2)}(x)$ yields

$$I_s^{(1)}(x)=I_s^{(2)}(-x)=e^{-ix}\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+s,-ix\right). \tag{42}$$

Substituting equations 42 and $\Phi^{(s)}(0)=(-1)^s s!$ back into equation 17 give the expansion

$$F_\nu(x)=-\cos(\pi\nu)\sum_{s=0}^\infty\frac{\Gamma\left(\frac{1}{2}(\nu+s+1)\right)}{\Gamma\left(\frac{1}{2}(\nu-s+1)\right)}\frac{2^s}{x^{s+1}}+\frac{1}{\sqrt{2\pi}}\sum_{s=0}^\infty\frac{\Gamma\left(\nu+s+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s\right)}{2^s s!\Gamma\left(\nu-s+\frac{1}{2}\right)x^{s+\frac{1}{2}}}$$
$$\times\left[e^{i\frac{\pi}{2}(s-v-1/2)}e^{-ix}\Gamma\left(\frac{1}{2}+s,-ix\right)+e^{-i\frac{\pi}{2}(s-v-1/2)}e^{ix}\Gamma\left(\frac{1}{2}+s,ix\right)\right]. \tag{43}$$

On its own it is not clear how to interpret equation 43 or how it is even useful numerically because it is composed of two terms expressed in two different scales.

We proceed by rewriting one term in the scale of the other so that the entire expansion $F_\nu(x)$ is expressed in one scale.

First, let us reduce the second term into the scale of the first term. Since the scale is power-type, we expect that we should be able to recover the known Poincare asymptotic expansion of the integral. We accomplish our goal by expanding each term of the second term asymptotically as $x \to \infty$. We need the asymptotic expansion of the incomplete gamma function, which is given by

$$\Gamma(a,z) = z^{a-1}e^{-z}\sum_{k=0}^{\infty}(-1)^k(1-a)_k\frac{1}{z^k}, \quad z\to\infty, \tag{44}$$

for $|\mathrm{Arg}z| < 3\pi/2$. Applying the asymptotic expansion 44 in the second term of equation 43 and interchanging the order of summations, we arrive, after some lengthy simplifications, at the Poincare asymptotic expansion 40. The Poincare asymptotic expansion for this case is exact; that is, there is no neglected subdominant term in the expansion.

Now we reverse the process by bringing the first term of equation 43 into the scale of the second term, that is rewriting the first term to combine with the second term. Let us consider in isolation the first term of equation 43. We apply the reflection formula to $\Gamma\left(\frac{1}{2}(v+s+1)\right)$ in the first term of equation 43. We split the resulting series in odd and even terms, sum them separately, and then combine their results again. After some lengthy simplifications, the first term assumes the form

$$-\pi\cot(\pi v)\sum_{k=0}^{\infty}\frac{2^k}{x^{k+1}}\frac{\left[e^{-i\frac{\pi}{2}v}(i)^{k+1}+e^{i\frac{\pi}{2}v}(-i)^{k+1}\right]}{\Gamma\left(\frac{1}{2}(1-v-k)\right)\Gamma\left(\frac{1}{2}(1+v-k)\right)}. \tag{45}$$

We are now in the position to transform the first term to combine with the second term. Let us consider the sum

$$\sum_{k=0}^{\infty}\frac{2^k}{x^{k+1}}\frac{i^k}{\Gamma\left(\frac{1}{2}(1-v-k)\right)\Gamma\left(\frac{1}{2}(1+v-k)\right)}, \tag{46}$$

which is proportional to the first term of equation 45. The idea is to re-expand the coefficients involving the gamma function. There are potentially infinitely many ways to do this, but we are constrained with our requirement that eventually the two terms of equation 43 must combine.

We accomplish our goal by using the following well-known hypergeometric sum,

$$F\left(a,1-a;b;\frac{1}{2}\right) = \frac{2^{1-b}\sqrt{\pi}\Gamma(b)}{\Gamma\left(\frac{1}{2}(a+b)\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)} \tag{47}$$

where F is the hypergeometric function $F(a,b;c;z) = \sum_{s=0}^{\infty}(a)_s(b)_s z^s/(c)_s s!$. Comparing $\Gamma\left(\frac{1}{2}(1-v-k)\right)\Gamma\left(\frac{1}{2}(1+v-k)\right)$ and $\Gamma\left(\frac{1}{2}(a+b)\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)$, we identify $a=\frac{1}{2}-\nu$ and $b=\frac{1}{2}-k$. Then we have

$$\frac{1}{\Gamma\left(\frac{1}{2}(1-v-k)\right)\Gamma\left(\frac{1}{2}(1+v-k)\right)} = \frac{F\left(\frac{1}{2}-v,\frac{1}{2}+v;\frac{1}{2}-k;\frac{1}{2}\right)}{2^{\frac{1}{2}+k}\sqrt{\pi}\Gamma\left(\frac{1}{2}-k\right)} \tag{48}$$

Substituting equation 48 back into equation 46, expanding the hypergeometric function and then interchanging the order of summations, the sum 46 becomes

$$\frac{1}{\sqrt{2\pi}}\sum_{s=0}^{\infty}(-1)^s\frac{\left(\frac{1}{2}-v\right)_s\left(\frac{1}{2}+v\right)_s}{2^s\, s!\, x}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{2}-s\right)_k\frac{1}{(-ix)^k}. \tag{49}$$

We have arrived at the most crucial step of bringing the two terms in one scale. We compare the inner sum 49 with the known asymptotic expansion of the incomplete gamma function, which is given equation 44. We find that the inner sum of the double series 49 is just proportional to the asymptotic expansion of the incomplete gamma function. With the appropriate identifications of the parameter a and z in equation 44, we replace the inner sum of equation 49 with the incomplete gamma function and then substitute back $T_\nu(x)$ into equation 65 to obtain

$$\begin{aligned} F_\nu(x) \;=\; & \csc(\nu\pi)\sum_{s=0}^{\infty}\frac{1}{(2x)^{s+\frac{1}{2}}}\frac{\left(\frac{1}{2}-v\right)_s\left(\frac{1}{2}+v\right)_s}{s!\left(\frac{1}{2}\right)_s} \\ & \times\left[e^{i\frac{\pi}{2}(s+v-\frac{3}{2})}e^{-ix}\Gamma\left(\frac{1}{2}+s,-ix\right)+e^{-i\frac{\pi}{2}(s+v-\frac{3}{2})}e^{ix}\Gamma\left(\frac{1}{2}+s,ix\right)\right], \end{aligned} \tag{50}$$

We have succeeded in rewriting the sum in terms of the scale given by the incomplete gamma function.

The expansions 40 and 50 are formally equivalent, one is just the rearrangement of the other. However, from the stand point of providing a numerical approximation of the value of integral, they are not equivalent. In Table-3 we compare the relative error of the optimally truncated expansions 40 and 50. We find, as in the previous two examples, that the distributional approach gives an asymptotic expansion that yield more accurate approximations than the Poincare asymptotic expansion. The steps leading to equation 50 gives us an idea why the resummed version of the Poincare asymptotic expansion is more accurate. In replacing the divergent inner sum of equation 49 with its corresponding sum (proportional to the incomplete gamma function), we have actually included the contribution of the tail of the asymptotic series, which is excluded by superasymptotic expansion of the Poincare asymptotic expansion. The expansion 50 then includes contributions from terms that are excluded by the Poincare asymptotic expansion.

In the previous two examples, the asymptotic expansion due to distributional approach are asymptotic sequences. This is not true for the present example. Consider the sequence

$$\varphi_s(x)=\frac{1}{x^{s+\frac{1}{2}}}\left[e^{i\frac{\pi}{2}(s+v-\frac{3}{2})}e^{-ix}\Gamma\left(\frac{1}{2}+s,-ix\right)+e^{-i\frac{\pi}{2}(s+v-\frac{3}{2})}e^{ix}\Gamma\left(\frac{1}{2}+s,ix\right)\right] \tag{51}$$

for $s=0,1,2,\dots$. From the leading order asymptotic behavior of the incomplete gamma function, $\Gamma(a,z)\sim z^{a-1}e^{-z}$ as $z\to\infty$, we have $\varphi_{s+j}(x)\sim 2\sin(\pi\nu/2)/x$

	Poincare		Resummed	
x	Relative error	Least term	Relative error	Least term
10	4.0×10^{-5}	11	1.1×10^{-9}	25
15	2.3×10^{-6}	15	3.1×10^{-13}	37
20	3.0×10^{-8}	21	1.3×10^{-17}	55
25	1.8×10^{-9}	25	1.5×10^{-21}	68
30	2.5×10^{-11}	31	8.8×10^{-26}	78
35	1.5×10^{-12}	35	3.1×10^{-30}	93
40	2.2×10^{-14}	41	7.4×10^{-34}	105
45	1.3×10^{-15}	45	1.0×10^{-38}	119
50	1.9×10^{-17}	51	7.7×10^{-42}	138

Table 3: Comparison of the relative error of the optimally truncated Poincare series and resummed series for $\nu = 1/3$.

as $x \to \infty$ for all $j = 0, 1, 2, \ldots$, so that we have the limit

$$\lim_{x\to\infty} \frac{\varphi_{s+j}(x)}{\varphi_s(x)} = 1, \quad j = 0, 1, 2, \ldots, \tag{52}$$

which implies that $\varphi_{s+j}(x) \neq o(\varphi_s(x))$ as $x \to \infty$ for all $j > 0$. Hence sequence 51 is not an asymptotic sequence nor a semiasymptotic one.

7 Conclusion

We have applied the distributional method of McClure and Wong in exactifying the Hankel integral. The proper treatment of divergent integrals arising from term by term integration using the theory of distribution has led to the recovery of the Poincare asymptotic expansion of the Hankel integral plus subdominant beyond all order terms. The treatment has yielded an asymptotic expansion that may give a finite series representation to a given Hankel integral, and an asymptotic expansion that provides numerical approximation to the value of the integral with spectacular accuracy.

Our results provide advances in two directions. First, it gives the possibility of direct construction of an exact asymptotic expansion in non-power-type scale which is already a transformation of an asymptotic expansion in power-type scale. In general, it is not difficult to see that the latter expansion (in non-power-type scale) is more accurate than the former expansion (in power-type scale): the fact that somewhere Poincare type asymptotic expansions arising in the intermediate steps from the power type to the non-power-type are being replaced by their closed form analytic versions already indicate that contributions from tails of Poincare asymptotic expansion, which are discarded in superasymptotic summation, are accounted for in the transformed version of the asymptotic expansion in power type scale. This is important as transformations into non-power-type scales may provide an alternative to hyperasymptotic summation, which has been developed to account for the contribution of the tail of Poincare type asymptotic expansions [38, 39, 40]. In hyperasymptotics, one needs repeated re-expansion of the remainder term to obtain more accurate approximation of the sum of the asymptotic expansion. On the other hand, the transformation to a non-power-type expansion may involve only one step, say, by direct application of the McClure and Wong method. As we have seen here, the one time optimal truncation of the non-power-type expansion is already very accurate; this accuracy may rival the accuracy of hyperasymptotics obtained by multiple optimal truncations and repeated re-expansion of the remainder terms. It is a distinct possibility that the expansion in non-power-type scale can be subjected to hyperasymptotics as well, provided, of course, remainder terms can be explicitly determined, which we lack in the present work and hope to address elsewhere. If it is possible to do hypeasymptotics on the transformed expansion, then it is expected that more accuracy can be wringed from the expansion. Since superasymptotic summation of the expansion is already accurate, adding more accuracy may already yield ultra-accurate sum of the divergent asymptotic series.

Second, the general theory of Poincare type asymptotic expansion is couched in terms of asymptotic scales [6, 7]. Despite the caveats and controversies in the general theory [6], generalized asymptotic expansions in asymptotic scales are made meaningful by the fact that their expansions in such scales are unique, and unambiguous order estimates can be made on their remainder terms [7]. This commands keeping the theory in its current form. However, we have seen from our last example that the distributional method has led to transforming the Poincare asymptotic expansion into a series in non-asymptotic scale. The series has behaved similarly with other asymptotic series as it demonstrated the typical behavior of a traditional asymptotic series: the terms initially decrease in magnitude and eventually increase without bound. Optimal truncation of this series yielded an accurate approximation of the Hankel integral; this shows that asymptotic series may actually be written meaningfully in non-asymptotic scale. This is important as it is well known that generalized asymptotic expansion in asymptotic scale may be useless either analytically or numerically [6]; expansions in non-asymptotic scales may then be even meaningful, at least numerically. Altogether our results clearly call for a re-examination of the concept of general asymptotic expansion. The numerical usefulness of the expansion that we have obtained involving non-asymptotic scale is a compelling motivation to develop further the general theory of asymptotic expansions.

Appendix A: Derivation of Equation 17

To derive the main result 17, we start with equation 12 which we rewrite here in the form

$$I(b) = \frac{1}{2}\sum_{s=0}^{\infty}\left[a_s^{(1)}\langle e_s^{(1)}, \Phi\rangle + a_s^{(2)}\langle e_s^{(2)}, \Phi\rangle\right] + \frac{1}{2}\sum_{s=0}^{\infty}(-1)^{s+1}\Phi^{(s)}(0)B_s \qquad (53)$$

where $B_s = b_s^{(1)} + b_s^{(2)}$, and $b_s^{(l)}$, for $l = 1$ and $l = 2$, is obtained from equation 9 by substituting the function $f(x)$ with $H_\nu^{(k)}(bx)$. In particular, the $b_s^{(l)}$'s are given by

$$b_s^{(l)} = \frac{(-1)^{s+1}}{s!} M\left[f^{(l)}; s+1\right] + \frac{(-1)^s}{s!} \sum_{k=0}^{s-1} a_k^{(l)} \frac{\Gamma\left(s-k+\frac{1}{2}\right)}{b^{s-k+\frac{1}{2}}} e^{(-1)^{l+1} i \frac{\pi}{2}\left(s-k+\frac{1}{2}\right)}, \tag{54}$$

in which an empty sum is zero; the $a_k^{(l)}$'s are given by equation 11. Then

$$\begin{aligned} B_s &= \frac{(-1)^{s+1}}{s!} M\left[f^{(1)} + f^{(2)}, s+1\right] \\ &\quad +2\sqrt{\frac{2}{\pi}} \frac{(-1)^s}{s!} \frac{\cos\left(\frac{\pi}{2}(s-\nu)\right)}{b^{s+1}} \sum_{k=0}^{s-1} \frac{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k k!\, \Gamma\left(\nu-k+\frac{1}{2}\right)} \end{aligned} \tag{55}$$

The second term of equation 53 can now be computed to give

$$\begin{aligned} \frac{1}{2}\sum_{s=0}^{\infty}(-1)^{s+1}\Phi^{(s)}(0) B_s &= \frac{1}{2}\sum_{s=0}^{\infty} \frac{\Phi^{(s)}(0)}{s!} M\left[f^{(1)} + f^{(2)}; s+1\right] \\ &\quad -\sqrt{\frac{2}{\pi}} \sum_{s=0}^{\infty} \frac{\Phi^{(s)}(0)}{s!} \frac{\cos\left(\frac{\pi}{2}(s-\nu)\right)}{b^{s+1}} \sum_{k=0}^{s-1} \frac{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k k! \Gamma\left(\nu-k+\frac{1}{2}\right)}. \end{aligned} \tag{56}$$

Before we obtain an explicit expression for the first term of equation 53, let us first discuss how we arrive at equation 13. From equation 6, we have

$$\langle e_s^{(l)}, \Phi\rangle = \frac{1}{s!}\int_0^\infty \Phi^{(s+1)}(x) \int_x^\infty (\tau - x)^s e_s^{(l)}(\tau)\, \mathrm{d}\tau \mathrm{d}x,\ l = 1, 2. \tag{57}$$

We integrate in x by integration by parts with $v = \int_x^\infty (\tau - x)^s e_s^{(l)}(\tau)\mathrm{d}\tau$ and $\mathrm{d}u = \Phi^{(s+1)}(x)\mathrm{d}x$; we perform the integration by parts repeatedly in the same manner until we reach integration in $\Phi(x)$ alone. In the process we encounter integrals of the form $\int_0^\infty \tau^{s-m} e_s^{(l)}(\tau)\,\mathrm{d}\tau = \int_0^\infty \tau^{-m-\frac{1}{2}} e^{(-1)^{l+1} i b \tau}\,\mathrm{d}\tau$, for $m = 0, \dots, s$, which are divergent; these integrals are evaluated by analytic extension of the known integral $\int_0^\infty x^{-a} e^{-bx}\mathrm{d}x = b^{-1+a}\Gamma(1-a)$, for $\mathrm{Re}(b) > 0$ and $\mathrm{Re}(a) < 1$. Then repeated integration by parts, coupled with analytic extension, leads to equation 13.

Now substituting equations 13 back into the first term of equation 53 and performing a shift in the index of summation yield

$$\begin{aligned} &\frac{1}{2}\sum_{s=0}^{\infty}\left[a_s^{(1)}\langle e_s^{(1)}, \Phi\rangle + a_s^{(2)}\langle e_s^{(2)}, \Phi\rangle\right] = \\ &\quad \frac{1}{2}\sqrt{\frac{2}{\pi}} \sum_{s=0}^{\infty} \frac{\Gamma\left(\nu+s+\frac{1}{2}\right)}{2^s s! \Gamma\left(\nu-s+\frac{1}{2}\right) b^{s+\frac{1}{2}}} \left[e^{i\frac{\pi}{2}\left(s-\nu-\frac{1}{2}\right)} I_s^{(1)} + e^{-i\frac{\pi}{2}\left(s-\nu-\frac{1}{2}\right)} I_s^{(2)}\right] \\ &\quad -\sqrt{\frac{2}{\pi}} \sum_{s=0}^{\infty} \frac{\Phi^{(s)}(0)\cos\left(\frac{\pi}{2}(s-\nu)\right)}{s! b^{s+1}} \sum_{k=s}^{\infty} \frac{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k k! \Gamma\left(\nu-k+\frac{1}{2}\right)} \end{aligned} \tag{58}$$

Notice that the inner sums of the second terms of equations 56 and 58 combine into a single series. Adding equations 56 and 58 then yield

$$\begin{aligned} I(b) &= \frac{1}{2}\sum_{s=0}^{\infty}\frac{\Phi^{(s)}(0)}{s!}\frac{\Gamma\left(\frac{1}{2}(\nu+s+1)\right)}{\Gamma\left(\frac{1}{2}(\nu-s+1)\right)}\left(\frac{2}{b}\right)^{s+1} \\ &+\frac{1}{\sqrt{2\pi}}\sum_{s=0}^{\infty}\frac{\Gamma\left(\nu+s+\frac{1}{2}\right)}{2^s s!\Gamma\left(\nu-s+\frac{1}{2}\right)b^{s+\frac{1}{2}}}\left[e^{i\frac{\pi}{2}(s-\nu-\frac{1}{2})}I_s^{(1)}+e^{-i\frac{\pi}{2}(s-\nu-\frac{1}{2})}I_s^{(2)}\right] \\ &-\sqrt{\frac{2}{\pi}}\sum_{s=0}^{\infty}\frac{\Phi^{(s)}(0)}{s!}\frac{\cos\left(\frac{\pi}{2}(s-\nu)\right)}{b^{s+1}}\sum_{k=0}^{\infty}\frac{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k k!\Gamma\left(\nu-k+\frac{1}{2}\right)} \qquad (59)\end{aligned}$$

where we have already evaluated the indicated Mellin transform in the second term, given by equation 15.

The inner sum in the third term of equation 59 can be explicitly evaluated by means of hypergeometric functions; in particular, we have the sum

$$\sum_{k=0}^{\infty}\frac{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k k!\Gamma\left(\nu-k+\frac{1}{2}\right)}=\Gamma\left(s+\frac{1}{2}\right){}_2F_1\left(\frac{1}{2}-\nu,\frac{1}{2}+\nu;\frac{1}{2}-s;\frac{1}{2}\right). \qquad (60)$$

The hypergeometric function can be further processed by using the well known identity [?, p.387,no.15.14.30]

$${}_2F_1\left(a,1-a;c;\frac{1}{2}\right)=\frac{2^{1-c}\sqrt{\pi}\Gamma(c)}{\Gamma\left(\frac{1}{2}(a+c)\right)\Gamma\left(\frac{1}{2}(c-a+1)\right)}. \qquad (61)$$

We identify $a=(1/2-\nu)$ and $c=(1/2-s)$. Then we have the sum

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k k!\Gamma\left(\nu-k+\frac{1}{2}\right)} &= \frac{(-1)^s\sqrt{2\pi}\,\pi}{\Gamma\left(\frac{1}{2}(1-\nu-s)\right)\Gamma\left(\frac{1}{2}(\nu-s+1)\right)} \\ &= (-1)^s\sqrt{2\pi}\sin\left(\frac{\pi}{2}(\nu+s+1)\right)\frac{\Gamma\left(\frac{1}{2}(\nu+s+1)\right)}{\Gamma\left(\frac{1}{2}(\nu-s+1)\right)} \qquad (62)\end{aligned}$$

Equation 62 is obtained by employing the reflection formula for the gamma function in the first line. Substituting equation 62 back into equation 59 gives equation 3 and explicitly yields the exactifying term given by equation 16. Now using the identity $\cos(\pi(s-\nu)/2)\sin(\pi(\nu+s+1)/2)=((-1)^s+\cos(\pi\nu))/2$, finally yields equation 17.

Appendix B: Derivation of Equation 32 by Mellin-Barnes Method

Here we summarize how the asymptotic expansion 32 is obtained by means of Mellin-Barnes integral method. The Mellin-Barnes method rewrites the integral $I(x)=\int_0^{\infty}f(t)h(xt)\,\mathrm{d}t$ into the form $I(x)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}x^{-z}M[f,1-$

$z]\, M[h;z]\, \mathrm{d}z$, where $M[f,z] = \int_0^\infty t^{z-1} f(t) \mathrm{d}t$ is the Mellin transform of $f(t)$ and c is some real constant. This is obtained by taking the Mellin transform of both sides of the integral and then inverting the the resulting expression. Applying this to the Hankel transform yields

$$I_1(b) = \frac{a}{2c} e^{-a^2/4c} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \left(\frac{b^2}{4c}\right)^{-z} M\left(z + \frac{1}{2}, \frac{3}{2}, \frac{a^2}{4c}\right) \Gamma(z)\ \mathrm{d}z \tag{63}$$

where $M(a;b;s)$ is the Kummer function $M(a,b,s) = \sum_{k=0}^{\infty} (a)_k s^k/(b)_k k!$ and for any $d > 0$. Equation 63 follows from using the integral

$$\int_0^\infty t^{-z} e^{-ct^2} \sin(at)\, \mathrm{d}t = \frac{a}{2c} c^{z/2} \Gamma\left(1 - \frac{z}{2}\right) M\left(1 - \frac{z}{2}, \frac{3}{2}, -\frac{a^2}{4c}\right) \quad \mathrm{Re}\, z < 2, \tag{64}$$

followed by using the reflection formula $M(a,b,s) = e^s M(a,b,s)$; this integral is obtained by expanding the sine and performing a term by term integration. We point out that the integral of the left hand side of 64 is a special case of the tabulated Mellin transform [41, p. 318, no.10]; however, the tabulated formula is not correct.

The integrand in 63 has no pole to the right of the complex plane. Translating the contour of integration to the right will not give any information on the asymptotic behavior of the integration. To obtain the asymptotic expansion, we introduce the power series expansion of the Kummer function in the integral 63; the result is

$$\begin{aligned} I_1(b) \quad &= \quad \frac{a}{2c} e^{-a^2/4c} \sum_{k=0}^{\infty} \frac{1}{(3/2)_k k!} \left(\frac{a^2}{4c}\right)^k \\ &\times \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \left(\frac{b^2}{4c}\right)^{-z} \frac{\Gamma(z + 1/2 + k)\Gamma(z)}{\Gamma(z + 1/2)}\, \mathrm{d}z \end{aligned} \tag{65}$$

where the identity $(a)_n = \Gamma(a+n)/\Gamma(n)$ has been used. Now the value of the integral is not dependent on d. This allows us to arbitrarily push the contour of integration to the far right. Under this condition the following identity has been established in [20, 7],

$$\begin{aligned} \frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+c)} \quad &= \quad \sum_{j=0}^{M-1} (-1)^j \frac{(c-a)_j (c-b)_j}{j!} \Gamma(s+a+b-c-j) \\ &\times \rho_M(s) \Gamma(s+a+b-c-M) \end{aligned} \tag{66}$$

where $\rho_M(s) = O(1)$ as $s \to \infty$.

Applying 66 in the integral in 65 and using the identity $\int_{d-i\infty}^{d+i\infty} \Gamma(s-\alpha) z^{-s}\, \mathrm{d}s = z^{-\alpha} e^{-z}$ lead to

$$I_1(b) = \frac{a}{2c} e^{-(a^2+b^2)/4c} \sum_{j=0}^{\infty} \frac{(-1)^j (1/2)_j}{j!} G_j\left(\frac{ab}{4c}\right) \left(\frac{b^2}{4c}\right)^{-j} \tag{67}$$

where

$$G_j(\gamma) = \sum_{k=0}^{\infty} \frac{(-k)_j}{k!(3/2)_k} \gamma^{2k} \tag{68}$$

The first few values are

$$\begin{aligned} G_0(\gamma) &= \frac{1}{2\gamma}\sinh(2\gamma) \\ G_1(\gamma) &= -\frac{1}{2}\cosh(2\gamma) + \frac{1}{4\gamma}\sinh(2\gamma) \\ G_2(\gamma) &= -\frac{6}{8}\cosh(2\gamma) + \frac{1}{8\gamma}(3+4\gamma^2)\sinh(2\gamma) \\ G_3(\gamma) &= -\frac{2}{16}(15+4\gamma^2)\cosh(2\gamma) + \frac{3}{16\gamma}(5+8\gamma^2)\sinh(2\gamma) \end{aligned} \tag{69}$$

Substituting these back into 67 and rearranging the resulting series in increasing order of reciprocal powers of the asymptotic parameter b reproduces equation 33.

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